Let $u_a(t) = \begin{cases} 1, & \text{if } t \geqslant a, \\ 0, & \text{if } t < a, \end{cases}$ be the unit step function[32].

Example 176. $$\mathcal{L}(u_a(t)) = \int_0^\infty e^{-st} u_a(t) \mathrm{d}t = \int_a^\infty e^{-st} \mathrm{d}t = \left[-\frac{e^{-st}}{s} \right]_{t=a}^\infty = \frac{e^{-sa}}{s}.$$ ♢

Example 177. Note that $u_a(t) f(t-a)$ is $f(t)$ delayed by a (make a sketch!). We find

$$\mathcal{L}(u_a(t) f(t-a)) = \int_a^\infty e^{-st} f(t-a) \mathrm{d}t = \int_0^\infty e^{-s(\tilde{t}+a)} f(\tilde{t}) \mathrm{d}\tilde{t} = e^{-sa} F(s).$$ ♢

Example 178. What is $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+1} \right)$?

Solution. $\frac{1}{s+1}$ is the Laplace transform of e^{-t}. Hence, $\frac{e^{-2s}}{s+1}$ is the Laplace transform of e^{-t} delayed by 2. In other words, $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+1} \right) = u_2(t) e^{-(t-2)}$. ♢

The next example illustrates that any piecewise defined function can be written using a single equation involving step functions. This is based on the simple observation that $u_a(t) - u_b(t)$ is a function which is 1 on the interval $[a, b)$ but zero everywhere else.

Example 179. Consider $f(t) = \begin{cases} t^2, & \text{if } 0 \leqslant t \leqslant 1, \\ 1, & \text{if } 1 \leqslant t \leqslant 2, \\ \cos(t-2), & \text{if } t \geqslant 2. \end{cases}$

Then, $f(t) = t^2(u_0(t) - u_1(t)) + 1(u_1(t) - u_2(t)) + \cos(t-2) u_2(t)$.

It is left as an exercise to compute the Laplace transform of $f(t)$ from here. Note that, for instance, to find $\mathcal{L}(t^2 u_1(t))$, we want to use $\mathcal{L}(u_a(t) f(t-a)) = e^{-sa} F(s)$ with $a = 1$ and $f(t-1) = t^2$; then, $f(t) = (t+1)^2 = t^2 + 2t + 1$ has Laplace transform $F(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$, and we combine to get $\mathcal{L}(t^2 u_1(t)) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \right)$. ♢

Example 180. Solve the IVP $x'' + 3x' + 2x = f(t)$, $x(0) = x'(0) = 0$ with $f(t) = \begin{cases} 1, & t \in [3, 4], \\ 0, & \text{otherwise.} \end{cases}$

Solution. First, we write $f(t) = u_3(t) - u_4(t)$. We can now take the Laplace transform of the DE to get

$$s^2 X(s) - s x(0) - x'(0) + 3(s X(s) - x(0)) + 2X(s) = \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s} = (e^{-3s} - e^{-4s}) \frac{1}{s}.$$

Using that $s^2 + 3s + 2 = (s+1)(s+2)$, we find

$$X(s) = (e^{-3s} - e^{-4s}) \frac{1}{s(s+1)(s+2)} = (e^{-3s} - e^{-4s}) \left[\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \right],$$

where A, B, C are determined by partial fractions (and will be computed below). Taking the Laplace inverse of each of the six terms in this product, as in Example 178, we find

$$x(t) = A(u_3(t) - u_4(t)) + B(u_3(t) e^{-(t-3)} - u_4(t) e^{-(t-4)}) + C(u_3(t) e^{-2(t-3)} - u_4(t) e^{-2(t-4)}).$$

If preferred, we can express this as $x(t) = \begin{cases} 0, & \text{if } t \leqslant 3, \\ A + Be^{-(t-3)} + Ce^{-2(t-3)}, & \text{if } t \in [3, 4], \\ B(e^{-(t-3)} - e^{-(t-4)}) + C(e^{-2(t-3)} - e^{-2(t-4)}) & \text{if } t \geqslant 4. \end{cases}$

Finally, $A = \left.\frac{1}{(s+1)(s+2)}\right|_{s=0} = \frac{1}{2}$, $B = \left.\frac{1}{s(s+2)}\right|_{s=-1} = -1$, $C = \left.\frac{1}{s(s+1)}\right|_{s=-2} = \frac{1}{2}$. Check that these values make $x(t)$ a continuous function (as it should be for physical reasons!). ♢

32. The special case $u_0(t)$ is also known as the Heaviside function, after Oliver Heaviside who, among many other things, coined terms like conductance and impedance. Note that $u_a(t) = u_0(t-a)$.

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