Sketch of Lecture 46

Review. Laplace transform, go over last example ♢

Example 174. Solve the IVP $x'' - 3x' + 2x = e^t$, $x(0) = 0$, $x'(0) = 1$.

Solution. **(old style, outline)** The characteristic polynomial is $s^2 - 3s + 2 = (s-1)(s-2)$. Since there is duplication, we have to look for a particular solution of the form $x_p = ate^t$. To determine a, we need to plug into the DE (we find $a = -1$). Then, the general solution is $x(t) = ate^t + c_1e^t + c_2e^{2t}$, and the initial conditions determine c_1 and c_2 (we find $c_1 = -2$ and $c_2 = 2$).

Solution. **(Laplace style)**

$$\begin{aligned} \mathcal{L}(x''(t)) - 3\mathcal{L}(x'(t)) + 2\mathcal{L}(x(t)) &= \mathcal{L}(e^t) \\ s^2X(s) - sx(0) - x'(0) - 3(sX(s) - x(0)) + 2X(s) &= \frac{1}{s-1} \\ (s^2 - 3s + 2)X(s) &= 1 + \frac{1}{s-1} = \frac{s}{s-1} \\ X(s) &= \frac{s}{(s-1)^2(s-2)} \end{aligned}$$

To find $x(t)$, we again use partial fractions. $X(s) = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}$ with coefficients (why?!)

$$C = \left.\frac{s}{(s-1)^2}\right|_{s=2} = 2, \quad A = \left.\frac{s}{s-2}\right|_{s=1} = -1, \quad B = \left.\frac{\mathrm{d}}{\mathrm{d}s}\frac{s}{s-2}\right|_{s=1} = \left.\frac{-2}{(s-2)^2}\right|_{s=1} = -2.$$

Finally, $x(t) = \mathcal{L}^{-1}\left(\frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}\right) = Ate^t + Be^t + Ce^{2t} = -(t+2)e^t + 2e^{2t}$. ♢

Example 175. Solve the IVP $\boldsymbol{x}' = \begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix}\boldsymbol{x}$, $\boldsymbol{x} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

Solution. **(old style)** See Problem 2 on our practice problems for the final midterm exam. There we computed that

$$e^{At} = \begin{pmatrix} 3 - 2e^t & -1 + e^t \\ 6 - 6e^t & -2 + 3e^t \end{pmatrix}.$$

Hence, $\boldsymbol{x} = e^{At}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 + 2e^t \\ -2 + 6e^t \end{pmatrix}$.

Solution. **(Laplace style)** Writing $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, the IVP is equivalent to

$$\begin{aligned} x_1' &= -2x_1 + x_2, \quad x_1(0) = 1, \\ x_2' &= -6x_1 + 3x_2, \quad x_2(0) = 4. \end{aligned}$$

Taking the Laplace transform of both equations, we get

$$\begin{aligned} sX_1(s) - x_1(0) &= -2X_1(s) + X_2(s), \\ sX_2(s) - x_2(0) &= -6X_1(s) + 3X_2(s), \end{aligned}$$

or, equivalently,

$$\begin{aligned} (s+2)X_1(s) - X_2(s) &= 1, \\ 6X_1(s) + (s-3)X_2(s) &= 4. \end{aligned}$$

Adding $(s-3)$ times the first equation to the second one, we find $(6 + (s-3)(s+2))X_1(s) = s(s-1)X_1(s) = s+1$. Hence,

$$X_1(s) = \frac{s+1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}, \qquad A = \left.\frac{s+1}{s-1}\right|_{s=0} = -1, \quad B = \left.\frac{s+1}{s}\right|_{s=1} = 2.$$

Taking the inverse Laplace transform, it follows that $x_1(t) = -1 + 2e^t$. Similarly,

$$X_2(s) = (s+2)X_1(s) - 1 = \frac{(s+2)(s+1)}{s(s-1)} - 1 = \frac{4s+2}{s(s-1)} = \frac{C}{s} + \frac{D}{s-1}, \quad C = \left.\frac{4s+2}{s-1}\right|_{s=0} = -2, \; D = \left.\frac{4s+2}{s}\right|_{s=1} = 6,$$

which implies $x_2(t) = -2 + 6e^t$. In conclusion, $\boldsymbol{x} = \begin{pmatrix} -1 + 2e^t \\ -2 + 6e^t \end{pmatrix}$, as above. ♢

Armin Straub
astraub@illinois.edu