Review. Laplace transform, table from last class ◊

Example 168. $\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\mathrm{d}t = \int_0^\infty e^{-(s-a)t}f(t)\mathrm{d}t = F(s-a)$ ◊

Example 169. We also add the following to our table of Laplace transforms.

$$\mathcal{L}(tf(t)) = \int_0^\infty e^{-st}tf(t)\mathrm{d}t = \int_0^\infty -\frac{\mathrm{d}}{\mathrm{d}s}e^{-st}f(t)\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st}f(t)\mathrm{d}t = -F'(s)$$

In particular,

$$\begin{aligned}\mathcal{L}(t) &= \mathcal{L}(t\cdot 1) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s} = \frac{1}{s^2}\\ \mathcal{L}(t^2) &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2} = \frac{2}{s^3}\\ &\vdots\\ \mathcal{L}(t^n) &= \frac{n!}{s^{n+1}}.\end{aligned}$$

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Theorem 170. (Uniqueness of Laplace transforms) If $\mathcal{L}(f_1(t)) = \mathcal{L}(f_2(t))$, then $f_1(t) = f_2$.

At least for all t, for which $f_1(t)$ and $f_2(t)$ are continuous. (Note that redefining $f(t)$ at a single point, will not change its Laplace transform.)

Hence, we can recover $f(t)$ from $F(s)$. We write $\mathcal{L}^{-1}(F(s)) = f(t)$.

Example 171. If $F(s) = \frac{3s-7}{s^2+4}$, what is $f(t)$?

Solution. $F(s) = 3\frac{s}{s^2+2^2} - \frac{7}{2}\frac{2}{s^2+2^2}$. Hence, $f(t) = 3\cos(2t) - \frac{7}{2}\sin(2t)$. ◊

Example 172. If $F(s) = \frac{1}{(s-3)^2}$, what is $f(t)$?

Solution. $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right) = e^{3t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = te^{3t}$. ◊

Example 173. Solve the IVP $x'' - 3x' + 2x = e^{-t}$, $x(0) = 0$, $x'(0) = 1$.

Solution. (old style) The characteristic polynomial is $s^2 - 3s + 2 = (s-1)(s-2)$. Since there is no duplication, there is a particular solution $x_p = ae^{-t}$. To determine a, we compute $x_p'' - 3x_p' + 2x_p = 6ae^{-t} \overset{!}{=} e^{-t}$ and conclude $a = \frac{1}{6}$. The general solution thus is $x(t) = \frac{1}{6}e^{-t} + c_1e^t + c_2e^{2t}$. Solving $x(0) = \frac{1}{6} + c_1 + c_2 \overset{!}{=} 0$ and $x'(0) = -\frac{1}{6} + c_1 + 2c_2 \overset{!}{=} 1$, we find $c_2 = \frac{4}{3}$ and $c_1 = -\frac{3}{2}$. Hence, $x(t) = \frac{1}{6}e^{-t} - \frac{3}{2}e^t + \frac{4}{3}e^{2t}$.

Solution. (Laplace style)

$$\begin{aligned}\mathcal{L}(x''(t)) - 3\mathcal{L}(x'(t)) + 2\mathcal{L}(x(t)) &= \mathcal{L}(e^{-t})\\ s^2X(s) - sx(0) - x'(0) - 3(sX(s) - x(0)) + 2X(s) &= \frac{1}{s+1}\\ (s^2-3s+2)X(s) &= 1 + \frac{1}{s+1} = \frac{s+2}{s+1}\\ X(s) &= \frac{s+2}{(s-1)(s-2)(s+1)}\end{aligned}$$

To find $x(t)$, we use partial fractions to write $X(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}$. We find the coefficients as

$$A = \left.\frac{s+2}{(s-2)(s+1)}\right|_{s=1} = -\frac{3}{2}, \quad A = \left.\frac{s+2}{(s-1)(s+1)}\right|_{s=2} = \frac{4}{3}, \quad C = \left.\frac{s+2}{(s-1)(s-2)}\right|_{s=-1} = \frac{1}{6}.$$

Finally, $x(t) = \mathcal{L}^{-1}\left(\frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}\right) = Ae^t + Be^{2t} + Ce^{-t} = \frac{1}{6}e^{-t} - \frac{3}{2}e^t + \frac{4}{3}e^{2t}$, as above. ◊

Armin Straub
astraub@illinois.edu