Sketch of Lecture 43

Wed, 04/16/2014

Review: basic skills

We have learned quite a bit about complex numbers and linear algebra. These are also very useful for your general (mathematical) well-being outside of DEs. Here is a rough overview of what we got to know.

- We can calculate with (e.g. divide) complex numbers. Real and imaginary part.
- We are still amazed by Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$.
- Add and multiply vectors and matrices. Identity matrix.
- Compute determinants of matrices by row (or column, if you wish) expansion.

  The determinant is zero $\Longleftrightarrow$ the columns (or, equivalently, rows) are linearly dependent.
- Invert matrices (at least 2×2).
- Find eigenvalues λ of a matrix. These are the roots of the characteristic polynomial $\det(A-\lambda I)$.

  If the matrix is $n\times n$, then the characteristic polynomial is of degree n. Over the complex numbers there are always n roots/eigenvalues if we count with repetition.
- For each eigenvalue there is at least one eigenvector $\boldsymbol{v}$ and we know how to find it. If λ is a repeated, say m times, we may find up to m independent eigenvectors. If we find less, say only $k<m$, then λ is said to have defect $m-k$.
- If λ is defective, then we know that we can find generalized eigenvectors. These come in chains.
- We know how to take the exponential of a matrix: e^A

  How was e^A defined? Well, there is options... what is your favourite definition of e^a when a is just a number?
  Definition via Taylor series: $e^a = 1 + a + \frac{a^2}{2} + \frac{a^3}{6} + \ldots$ works just as well for matrices $e^A = 1 + A + \frac{A^2}{2} + \frac{A^3}{6} + \ldots$
  Via derivative: e^{at} is unique $x(t)$ such that $x' = ax$, $x(0) = 1$ vs. e^{At} is unique $\Phi(t)$ such that $\Phi' = A\Phi$, $\Phi(0) = I$

Review: systems of DEs

We spent basically all the time since the last midterm on systems of DEs. Here is a reminder why and where we got.

- Any high-order DE can be transformed into a first-order system.

  That's why we have been studying $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}$ for so long. It is not some esoteric special case that happens to be doable—far from that: any linear DE can be written in this form!! [And any DE can be approximated by a linear DE.]
- For linear systems $\boldsymbol{x}' = A(t)\boldsymbol{x}$ existence and uniqueness of solutions is for free.

  ... on the interval I where the entries of $A(t)$ are continuous.
- We are familiar with the Wronskian and fundamental matrices.

  The matrix exponential e^{At} is a particularly nice fundamental matrix. If $\Phi(t)$ is some fundamental matrix, then $e^{At} = \Phi(t)\Phi(0)^{-1}$.
- We can solve all homogeneous equations $\boldsymbol{x}' = A\boldsymbol{x}$ where A has constant entries.
  - First, find eigenvalues λ. For each λ, we then determine the eigenvectors. If λ turns out to be defective, then we have to look for generalized eigenvectors.
  - Here's a reminder how to get solutions out of a chain $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ of generalized eigenvectors for λ:

$$\begin{aligned}
(A-\lambda I)\boldsymbol{v}_1 &= 0 && \text{solution: } \boldsymbol{v}_1 e^{\lambda t}\\
(A-\lambda I)\boldsymbol{v}_2 &= \boldsymbol{v}_1 && \text{solution: } (\boldsymbol{v}_1 t + \boldsymbol{v}_2)e^{\lambda t}\\
&\vdots\\
(A-\lambda I)\boldsymbol{v}_k &= \boldsymbol{v}_{k-1} && \text{solution: } \left(\boldsymbol{v}_1\frac{t^{k-1}}{(k-1)!} + \boldsymbol{v}_2\frac{t^{k-2}}{(k-2)!} + \ldots \boldsymbol{v}_{k-1}t + \boldsymbol{v}_k\right)e^{\lambda t}.
\end{aligned}$$

  - If $\lambda = a + bi$ is a complex eigenvalue, then it occurs together with its conjugate $a - bi$. We can get real-valued solutions by taking real and imaginary part of the complex solutions.

    We only need to do that for one of $a \pm bi$ because the other will give rise to equivalent solutions.
- We learned how to solve inhomogeneous equations $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}(t)$.
  - If $\boldsymbol{x}_p(t)$ is some particular solution, then $\boldsymbol{x}_p(t) + \boldsymbol{x}_c(t)$ is the general solution.

    Here, $\boldsymbol{x}_c(t)$ denotes the general solution of the complementary equation $\boldsymbol{x}' = A\boldsymbol{x}$.
  - We know two methods to find an $\boldsymbol{x}_p(t)$: undetermined coefficients and variation of constants. Variation of constants, that is $\Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t$, can always to be used, whereas undetermined coefficients requires $\boldsymbol{f}(t)$ to be a linear combination of polynomials times exponentials (so that we can attach a "root" to it).

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