Inverting matrices of any size

In order to compute A^{-1}, we need to find a matrix X such that $AX = I$. If this equation has a solution X, then $X = A^{-1}$. As we have done before, we write $A|I$ and perform elimination on the rows. Instead of stopping at a triangular shape, we continue until we get $I|B$. Then $A^{-1} = B$. This is best explained by an example (which we can already do).

Example 160. Let $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$. Find A^{-1}.

Solution. We eliminate

$$\left.\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right. \underset{\substack{\frac{1}{3}r_1 \\ r_2 - \frac{2}{3}r_1}}{\Longrightarrow} \left.\begin{array}{cc|cc} 1 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & -2/3 & 1 \end{array}\right. \underset{\substack{r_1 - r_2 \\ 3r_2}}{\Longrightarrow} \left.\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 3 \end{array}\right. \Longrightarrow A^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}.$$

Solution. Using $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, we again find $A^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$. ◇

Example 161. Find e^{At} if $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}$.

Solution. We first compute a fundamental matrix $\Phi(t)$ for $\boldsymbol{x}' = A\boldsymbol{x}$. To begin with, we easily see that the eigenvalues are $\lambda = 1, 1, 2$ (why?!).

$\boldsymbol{\lambda = 2.}$ $\begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}\boldsymbol{v} = 0$. We find the eigenvector $\boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\boldsymbol{\lambda = 1.}$ $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}\boldsymbol{v} = 0$. We find the eigenvector $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ but no second independent eigenvector. $\lambda = 1$ has defect 1. We therefore solve $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and find, for instance, $\boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$.

Taken together, we have found that $\Phi(t) = \begin{pmatrix} e^t & te^t & 0 \\ 0 & \frac{1}{2}e^t & 0 \\ 0 & \frac{1}{2}e^t & e^{2t} \end{pmatrix}$ is a fundamental matrix.

We can now find e^{At} from $e^{At} = \Phi(t)\Phi(0)^{-1}$.

$$\Phi(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}, \quad \left.\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1 & 0 & 0 & 1 \end{array}\right. \underset{\substack{2r_2 \\ r_3 - r_2}}{\Longrightarrow} \left.\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right., \quad \Phi(0)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Hence,

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{pmatrix} e^t & te^t & 0 \\ 0 & \frac{1}{2}e^t & 0 \\ 0 & \frac{1}{2}e^t & e^{2t} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} e^t & 2te^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t - e^{2t} & e^{2t} \end{pmatrix}.$$

Solution. **(failed attempt)** We can write[31] $A = D + N$ with $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$. We quickly check that N is nilpotent: in fact, $N^2 = 0$. Therefore,

$$e^{Dt}e^{Nt} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}(I + Nt) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 1 & 2t & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{pmatrix} = \begin{pmatrix} e^t & 2te^t & 0 \\ 0 & e^t & 0 \\ 0 & -te^{2t} & e^{2t} \end{pmatrix}.$$

This cannot be e^{At} because of the entry $-te^{2t}$. What went wrong?! Well, in order to use $e^{At} = e^{Dt}e^{Nt}$ we first need to check that $DN = ND$. This is not the case here! ◇

31. Here is a twist on our usual approach, which can be used here: write $A = I + B$. Then B is not nilpotent, but we observe that $B^2 = B^3 = B^4 = \ldots$. Do you see how to use this to compute e^{Bt}, and then e^{At}?

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