There was nothing special about 2π-periodic functions considered last time (except that $\cos(t)$ and $\sin(t)$ have period 2π). Note that $\cos(\pi t/L)$ has period $2L$.

Theorem 151. Every* $2L$-periodic function f can be written as a Fourier series

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{L}\right)+b_n\sin\left(\frac{n\pi t}{L}\right)\right).$$

Technical detail*: f needs to be, e.g., piecewise smooth.
Also, if t is a discontinuity, then the Fourier series converges to the average $\frac{f(t^-)+f(t^+)}{2}$.

The Fourier coefficients a_n, b_n are unique and can be computed as

$$a_n=\frac{1}{L}\int_{-L}^{L}f(t)\cos\left(\frac{n\pi t}{L}\right)\mathrm{d}t,\qquad b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t.$$

Review. Last time, we computed $f(t)=\begin{cases}-1, & \text{for } t\in(-\pi,0),\\ +1, & \text{for } t\in(0,\pi),\\ 0, & \text{for } t=-\pi,0,\pi\end{cases}=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\sin(nt).$ ◇

Example 152. Find the Fourier series of the 2-periodic function $g(t)=\begin{cases}-1 & \text{for } t\in(-1,0)\\ +1 & \text{for } t\in(0,1)\\ 0 & \text{for } t=-1,0,1\end{cases}$.

Solution. Instead of computing from scratch, we can use the fact that $g(t)=f(\pi t)$, with f as reviewed above, to get $g(t)=f(\pi t)=\sum_{n\text{ odd}}\frac{4}{\pi n}\sin(n\pi t)$. ◇

Remark 153. Convergence of such series is not obvious! Recall, for instance, that the (odd part of) the harmonic series $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots$ diverges. ◇

Theorem 154. If $f(t)$ is continuous and $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{L}\right)+b_n\sin\left(\frac{n\pi t}{L}\right)\right)$, then* $f'(t)=\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}b_n\cos\left(\frac{n\pi t}{L}\right)-\frac{n\pi}{L}a_n\sin\left(\frac{n\pi t}{L}\right)\right)$ (i.e., we can differentiate termwise).

Technical detail*: f' needs to be, e.g., piecewise smooth (so that it has a Fourier series itself).

Example 155. Let $h(t)$ be the 2-periodic function with $h(t)=\begin{cases}-t & \text{for } t\in(-1,0)\\ +t & \text{for } t\in(0,1)\end{cases}$. Compute the Fourier series of $h(t)$.

Solution. We could just use the integral formulas to compute a_n and b_n. Since $h(t)$ is even (plot it!), we will find that $b_n=0$. Computing a_n is left as an exercise.

Solution. Note that $h(t)$ is continuous and $h'(t)=g(t)$, with $g(t)$ as in Example 152. Hence, we can apply Theorem 154 to conclude

$$h'(t)=g(t)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\sin(n\pi t)\quad\Longrightarrow\quad h(t)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\left(-\frac{1}{\pi n}\right)\cos(n\pi t)+C,$$

where $C=\frac{a_0}{2}=\frac{1}{2}\int_{-1}^{1}h(t)\mathrm{d}t$ is the constant of integration. Thus, $h(t)=\frac{1}{2}-\sum_{n\text{ odd}}\frac{4}{\pi^2n^2}\cos(n\pi t)$. ◇

Remark 156. Note that $t=0$ in the last Fourier series, gives us $\frac{\pi^2}{8}=\frac{1}{1}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$. As an exercise, you can try to find from here the fact that $\sum_{n\geqslant 1}\frac{1}{n^2}=\frac{\pi^2}{6}$. Similarly, we can use Fourier series to find that $\sum_{n\geqslant 1}\frac{1}{n^4}=\frac{\pi^4}{90}$. JFF: if you recall from lecture 13, these are the values $\zeta(2)$ and $\zeta(4)$ of the Riemann zeta function $\zeta(s)$. No such values are known for $\zeta(3),\zeta(5),\ldots$ Nobody believes these to be rational numbers, but it was only in 1978 that Apéry proved that $\zeta(3)$ is not a rational number[29]. ◇

Example 157. The function $g(t)$, from in Example 152, is not continuous. For all values, except the discontinuities, we have $g'(t)=0$. On the other hand, differentiating the Fourier series termwise, results in $4\sum_{n\text{ odd}}\cos(n\pi t)$, which diverges[30] for most values of t (that's easy to check for $t=0$). This illustrates that we cannot apply Theorem 154 because of the missing continuity. ◇

29. We also know that at least one of $\zeta(5),\zeta(7),\zeta(9),\zeta(11)$ is not a rational number. (Our state of ignorance!)

30. The issues we are facing here can be fixed by generalizing the notion of function to distributions. (Maybe you have heard of the Dirac delta "function".)

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