Sketch of Lecture 34

Review. If $\boldsymbol{x}' = A\boldsymbol{x}$, with A an $n \times n$ matrix (with constant entries), then n independent solutions $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ can be combined into a fundamental matrix $\Phi = (\boldsymbol{x}_1 \ldots \boldsymbol{x}_n)$.

- The Wronskian is $W(t) = \det \Phi$.
- The general solution is simply $\Phi \boldsymbol{c}$.
- $\Phi(t)$ satisfies the matrix equation $\Phi' = A\Phi$.

  This is a consequence of how matrix multiplication works, and a good test of your understanding. Indeed, $\Phi' = (\boldsymbol{x}_1' \ldots \boldsymbol{x}_n')$ and $A\Phi = A(\boldsymbol{x}_1 \ldots \boldsymbol{x}_n) = (A\boldsymbol{x}_1 \ldots A\boldsymbol{x}_n)$. ♢

Example 131. Find a fundamental matrix for $\boldsymbol{x}' = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \boldsymbol{x}$.

Solution. **(using e^{At})** Let $A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} = 2I + N$ with $N = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$.

We want to use $e^{At} = e^{2It + Nt} = e^{2It} e^{Nt}$. This is indeed possible though we have to be careful: $e^{A+B} = e^A e^B$ holds if $AB = BA$. The identity matrix commutes with every other matrix, so we are good here.

e^{2It} is simple to compute (so is the exponential of any diagonal matrix): $e^{2It} = e^{2t} I$.

e^{Nt} is also simple to compute because N is nilpotent: $N^2 = \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $N^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$$e^{Nt} = I + Nt + N^2 \frac{t^2}{2} + N^3 \frac{t^3}{6} + \ldots = I + Nt + N^2 \frac{t^2}{2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -t & t \\ 0 & 0 & 3t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{3}{2}t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -t & t - \frac{3}{2}t^2 \\ 0 & 1 & 3t \\ 0 & 0 & 1 \end{pmatrix}$$

Together, $e^{At} = e^{2t} \begin{pmatrix} 1 & -t & t - \frac{3}{2}t^2 \\ 0 & 1 & 3t \\ 0 & 0 & 1 \end{pmatrix}$ is a fundamental matrix.

Solution. **(using generalized eigenvectors)** The characteristic polynomial is $(2-\lambda)^3$. Hence, $\lambda = 2$ is the only eigenvalue and has multiplicity 3. Solving the eigenvector equation $\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, we (only) find $\boldsymbol{v}_1 = (1, 0, 0)^T$ (or any multiple). This means that $\lambda = 2$ has defect 2. There has to be a chain of length 3. To find a generalized eigenvector $\boldsymbol{v}_2$ of rank 2, we solve $\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and obtain, for instance, $\boldsymbol{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$. Similarly, to find a generalized eigenvector $\boldsymbol{v}_3$ of rank 3, we solve $\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ and obtain, for instance, $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ -1/3 \\ -1/3 \end{pmatrix}$.

The corresponding solutions are $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$, $\boldsymbol{x}_2 = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right] e^{2t} = \begin{pmatrix} t \\ -1 \\ 0 \end{pmatrix} e^{2t}$ and $\boldsymbol{x}_3 = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/3 \\ -1/3 \end{pmatrix} \right] e^{2t} = \begin{pmatrix} t^2/2 \\ -1/3 - t \\ -1/3 \end{pmatrix} e^{2t}$. Hence, $\begin{pmatrix} 1 & t & t^2/2 \\ 0 & -1 & -1/3 - t \\ 0 & 0 & -1/3 \end{pmatrix} e^{2t}$ is a(nother) fundamental matrix.

The fundamental matrices look different but they are equivalent. Check it! (We did.) ♢

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