Example 125. Consider $\boldsymbol{x}' = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix} \boldsymbol{x}$.

The characteristic polynomial $(1-\lambda)(7-\lambda)+9 = \lambda^2 - 8\lambda + 16 = (\lambda-4)^2$ has the double root $\lambda = 4$.
However, $\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} \boldsymbol{v} = 0$ has solution only $\boldsymbol{v} = c \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
We say that the eigenvalue 4 is defective with defect 1 (number of missing eigenvectors).
So far, we have found the solution $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$ but we are missing a second independent solution. ♢

We want to solve $\boldsymbol{x}' = A\boldsymbol{x}$. Suppose that λ is a repeated and defective eigenvalue.

- As a first attempt, we might try to look for a solution of the form $\boldsymbol{x} = \boldsymbol{w} t e^{\lambda t}$. Plugging into the DE, we get $\boldsymbol{x}' = \boldsymbol{w} e^{\lambda t} + \boldsymbol{w} \lambda t e^{\lambda t} \overset{!}{=} A\boldsymbol{x} = A\boldsymbol{w} t e^{\lambda t}$. Setting $t = 0$, this implies $\boldsymbol{w} = 0$ which means our first attempt failed.
- Not giving up, we next look for a solution of the form $\boldsymbol{x} = \boldsymbol{u} e^{\lambda t} + \boldsymbol{w} t e^{\lambda t}$. Plugging into the DE, we now get $\boldsymbol{x}' = \boldsymbol{u} \lambda e^{\lambda t} + \boldsymbol{w} e^{\lambda t} + \boldsymbol{w} \lambda t e^{\lambda t} \overset{!}{=} A\boldsymbol{x} = A\boldsymbol{u} e^{\lambda t} + A\boldsymbol{w} t e^{\lambda t}$. Equating coefficients, we find $A\boldsymbol{w} = \lambda \boldsymbol{w}$ and $A\boldsymbol{u} = \lambda \boldsymbol{u} + \boldsymbol{w}$. Equivalently, $(A - \lambda I)\boldsymbol{w} = 0$ and $(A - \lambda I)\boldsymbol{u} = \boldsymbol{w}$. [$\boldsymbol{u}$ will be called a generalized eigenvector of rank 2.]

Example. (cont'd) Let us find $\boldsymbol{u}$ such that $\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} \boldsymbol{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Note that the second equation is -1 times the first. Hence, setting $w_2 = c$, we get $w_1 = -c - \frac{1}{3}$. Any choice of c will give us a vector $\boldsymbol{w}$ that we need to construct a second solution. For instance, choosing $c = 0$, we get $\boldsymbol{w} = (-1/3, 0)^T$. This means that $\boldsymbol{x}_2 = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} -1/3 \\ 0 \end{pmatrix} \right] e^{4t}$ is a second independent solution of the DE. ♢

This approach works whenever we have a defective eigenvalue of multiplicity 2.

The same idea leads to the concept of generalized eigenvectors.

Definition 126. $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ form a chain of generalized eigenvectors for the eigenvalue λ if

$$\begin{aligned}
(A - \lambda I)\boldsymbol{v}_1 &= 0 && [\text{solution of } \boldsymbol{x}' = A\boldsymbol{x}\text{: } \boldsymbol{v}_1 e^{\lambda t}] \\
(A - \lambda I)\boldsymbol{v}_2 &= \boldsymbol{v}_1 && [\text{solution: } (\boldsymbol{v}_1 t + \boldsymbol{v}_2) e^{\lambda t}] \\
(A - \lambda I)\boldsymbol{v}_3 &= \boldsymbol{v}_2 && \left[\text{solution: } \left(\boldsymbol{v}_1 \frac{t^2}{2} + \boldsymbol{v}_2 t + \boldsymbol{v}_3\right) e^{\lambda t}\right] \\
&\vdots \\
(A - \lambda I)\boldsymbol{v}_k &= \boldsymbol{v}_{k-1} && \left[\text{solution: } \left(\boldsymbol{v}_1 \frac{t^{k-1}}{(k-1)!} + \boldsymbol{v}_2 \frac{t^{k-2}}{(k-2)!} + \ldots \boldsymbol{v}_{k-1} t + \boldsymbol{v}_k\right) e^{\lambda t}\right]
\end{aligned}$$

Some comments on generalized eigenvectors:

- Note that $\boldsymbol{v}_k$ satisfies $(A - \lambda I)^k \boldsymbol{v}_k = 0$ but $(A - \lambda I)^{k-1} \boldsymbol{v}_k = \boldsymbol{v}_1 \neq 0$. We say $\boldsymbol{v}_k$ is a generalized eigenvector of rank k.
- The vectors in several chains are independent if and only if the chains are based on independent eigenvectors (the $\boldsymbol{v}_1$'s).
- For every $n \times n$ matrix A, we can find n independent generalized eigenvectors. In particular, we can then find the general solution of $\boldsymbol{x}' = A\boldsymbol{x}$ by constructing the corresponding solutions as indicated above.

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