A crash course in linear algebra

Example 93. A typical 2×3 matrix is $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. It is composed of column vectors like $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and row vectors like $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$.

Matrices (and vectors) of the same dimensions can be added and multiplied by a scalar. For instance, $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 5 \\ 6 & 8 & 5 \end{pmatrix}$ or $3 \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$. $\diamond$

Remark 94. More generally, a vector space is an abstraction of a collection of objects that can be added and scaled: numbers, lists of numbers (like the above row and column vectors), arrays of numbers (like the above matrices), arrows, functions, polynomials, differential operators, solutions to homogeneous linear differential equations, ... $\diamond$

Example 95. The transpose A^T of a matrix A is the matrix obtained by interchanging the roles of rows and columns. For instance, $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$. $\diamond$

Example 96. Matrices of appropriate dimensions can also be multiplied.

This is based on the multiplication $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$ of row and column vectors.

For instance, $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}$. In general, we can multiply a $m \times n$ matrix A with a $n \times r$ matrix B to get a $m \times r$ matrix AB. Its entry in row i and column j is defined to be $(AB)_{ij} = (\text{row } i \text{ of } A) \begin{pmatrix} \text{column} \\ j \\ \text{of } B \end{pmatrix}$.

A good way to think about the multiplication $A\boldsymbol{x}$ is that the resulting vector is a linear combination of the columns of A with coefficients from $\boldsymbol{x}$. Similarly, we can think of $\boldsymbol{x}^T A$ as a combination of the rows of A.

Some nice properties of matrix multiplication are:

- There is a $n \times n$ identity matrix I (all entries are zero except the diagonal ones which are 1). It satisfies $AI = A$ and $IA = A$.
- The associative law $A(BC) = (AB)C$ holds. Hence, we can write ABC without ambiguity.
- The distributive laws including $A(B + C) = AB + AC$ hold. $\diamond$

Example 97. $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, so we have no commutative law. $\diamond$

The inverse A^{-1} of a matrix A is characterized by $A^{-1}A = I$ and $AA^{-1} = I$.

Example 98. You can check that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Useful to remember! $\diamond$

Example 99. Equations like $7x_1 - 2x_2 = 3$, $2x_1 + x_2 = 4$ can be equivalently expressed as $\begin{pmatrix} 7 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Multiplying (from the left!) by $\begin{pmatrix} 7 & -2 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{11} \begin{pmatrix} 1 & 2 \\ -2 & 7 \end{pmatrix}$ produces $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 1 & 2 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ which gives the solution of the original equations. $\diamond$

The determinant of A, written as $\det(A)$ or $|A|$, is a number with the property that:

$$\begin{aligned} \det(A) \neq 0 \quad &\Longleftrightarrow \quad A \text{ is invertible} \\ &\Longleftrightarrow \quad A\boldsymbol{x} = \boldsymbol{b} \text{ has a (unique) solution } \boldsymbol{x} \text{ (for all } \boldsymbol{b}) \\ &\Longleftrightarrow \quad A\boldsymbol{x} = 0 \text{ is only solved by } \boldsymbol{x} = 0 \end{aligned}$$

Example 100. $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ which appeared in the formula for the inverse. $\diamond$

We will compute determinants of larger matrices next time.

Armin Straub
astraub@illinois.edu