Inhomogeneous linear DEs

Recall that a linear DE is one of the form $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$. Writing[12] $D = \frac{\mathrm{d}}{\mathrm{d}x}$ and setting $L := D^n + p_{n-1}(x)\, D^{n-1} + \ldots + p_1(x) D + p_0(x)$, this DE takes the concise form $L \cdot y = f(x)$. [L is a linear differential operator.]

- Note that $L \cdot (y_1 + y_2) = L \cdot y_1 + L \cdot y_2$.

  In particular, if $L \cdot y_1 = 0$ and $L \cdot y_2 = 0$, then $L \cdot (y_1 + y_2) = 0$. Superposition!
  More generally, $L \cdot (C_1 y_1 + C_2 y_2) = C_1\, L \cdot y_1 + C_2\, L \cdot y_2$.

- Let y_p be a particular solution to $L \cdot y = f(x)$. Let $C_1 y_1 + \ldots + C_n y_n$ be the general solution of $L \cdot y = 0$. Then $y_p + C_1 y_1 + \ldots + C_n y_n$ is the general solution of $L \cdot y = f(x)$.

  Hence, solving an inhomogeneous linear DE reduces to two simpler problems!

Example 66. Find the general solution of $y'' + 4y = 12x$. *Hint*: $3x$ is a solution.

Solution. Here, $L = D^2 + 4$. We already know one solution $y_p = 3x$.
Solving $L \cdot y = 0$ gives $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$. [Make sure this is easy for you!]
Therefore, the general solution is $y_p + C_1 y_1 + C_2 y_2 = 3x + C_1 \cos(2x) + C_2 \sin(2x)$.

How to find the particular solution ourselves? Apply D^2 to the DE! We get $D^2(D^2 + 4) \cdot y = 0$, which is a homogeneous linear DE! Its general solution is $C_1 + C_2 x + C_3 \cos(2x) + C_4 \sin(2x)$. In particular, y_p is of this form for some choice of $C_1, \ldots, C_4$.

In fact, it simplifies our life to note that there has to be a particular solution of the simpler form $y_p = C_1 + C_2 x$. Why?!

Now, it only remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. $y_p'' + 4y_p = 4C_1 + 4C_2 x = 12x$ and we conclude $C_1 = 0$ and $C_2 = 3$. We found $y_p = 3x$, as used above. ◇

Example 67. Find the general solution of $y'' + 4y' + 4y = e^x$.

Solution. Here, $L = D^2 + 4D + 4 = (D+2)^2$. The general solution of $L \cdot y = 0$ is $(C_1 + C_2 x) e^{-2x}$.
Note that $(D-1) \cdot e^x = 0$. Hence, we apply $(D-1)$ to the DE to get $(D-1)(D+2)^2 \cdot y = 0$. This homogeneous linear DE has general solution $(C_1 + C_2 x) e^{-2x} + C_3 e^x$. We conclude that the original DE must have a particular solution $y_p = C_3 e^x$. To determine the value of C_3, we plug into the DE: $y_p'' + 4y_p' + 4y_p = 9C_3 e^x = e^x$. Hence, $C_3 = 1/9$. Finally, the general solution is $(C_1 + C_2 x) e^{-2x} + \frac{1}{9} e^x$. ◇

This method gives a recipe for solving nonhomogeneous linear DEs with constant coefficients.
It works whenever the right-hand side $f(x)$ is the solution of some homogeneous linear DE.

Theorem 68. To find a particular solution y_p of $L \cdot y = f(x)$.

- Let $r_1, \ldots, r_n$ be the (old) roots of the char poly of $L \cdot y = 0$.
- Let $s_1, \ldots, s_m$ be the (new) roots of the char poly of $L_{\text{rhs}} \cdot f = 0$, the HLDE (with constant coefficients) which $f(x)$ solves. (This is not possible for all $f(x)$.)
- It follows that y_p solves $L_{\text{rhs}}\, L \cdot y = 0$. Its char poly has roots $r_1, \ldots, r_n$, $s_1, \ldots, s_m$. Let $v_1, \ldots, v_m$ be the "new" solutions (i.e. not solutions of the "old" $L \cdot y = 0$). Now, we can find (unique) constants C_i such that $y_p = C_1 v_1 + \ldots + C_m v_m$.

Example 69. Find the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

Solution. "Old" roots $-2, -2$. "New" roots -2. Hence, there has to be a particular solution of the form $y_p = C x^2 e^{-2x}$. To find the value of C, we need to plug into the DE.
$y_p' = C(-2x^2 + 2x) e^{-2x}$. $y_p'' = C(4x^2 - 8x + 2) e^{-2x}$. Hence, $y_p'' + 4y_p' + 4y_p = 2C e^{-2x} \overset{!}{=} 7e^{-2x}$. $C = 7/2$.
Since $y_p = \frac{7}{2} x^2 e^{-2x}$, the general solution is $\left(C_1 + C_2 x + \frac{7}{2} x^2\right) e^{-2x}$. ◇

12. As in the proof of Theorem 42.