Review. population models ◇

Example 35. Short outbreaks of diseases among a population of constant size N.

Model the population as consisting of $S(t)$ susceptible, $I(t)$ infected and $R(t)$ recovered individuals ($N = S(t) + I(t) + R(t)$). In the SIR model,

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I,$$

with γ modeling the recovery rate and β the infection rate. This is a system of differential equations, something we will study in due time.

By the way, the following variation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I R, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I R,$$

which assumes "infectious recovery", was recently used to predict that facebook might loose 80% of its users by 2017. It's that claim, not mathematics (or even the modeling), which attracted a lot of media attention[6]. ◇

Homogeneous linear DEs with constant coefficients

Review. linear higher-order differential equations ◇

Example 36. Find the general solution of $y'' - y' - 2y = 0$. *Hint:* Try solutions e^{rx}.

Solution. Plugging e^{rx} into the DE, we get $r^2 e^{rx} - r e^{rx} - 2e^{rx} = 0$.

Equivalently, $r^2 - r - 2 = 0$. This is called the characteristic equation. Its solutions are $r = 2, -1$.

This means we found the two solutions $y_1 = e^{2x}$, $y_2 = e^{-x}$.

By superposition, the general solution[7] is $y = A e^{2x} + B e^{-x}$. ◇

This approach applies to any homogeneous linear DE with constant coefficients! If the characteristic equation has enough different roots, then we find the general solution. [by superposition!]

Example 37. Find the general solution of $y''' + 7y'' + 14y' + 8y = 0$.

Solution. The characteristic equation is $r^3 + 7r^2 + 14r + 8 = (r+1)(r+2)(r+4)$.

Hence, we found the solutions $y_1 = e^{-x}$, $y_2 = e^{-2x}$, $y_3 = e^{-4x}$. That's enough (independent!) solutions for a third-order DE. By superposition, the general solution is $y(x) = A e^{-x} + B e^{-2x} + C e^{-4x}$. ◇

6. http://blogs.wsj.com/digits/2014/01/22/controversial-paper-predicts-facebook-decline/

7. Well, this is certainly a two-parameter family of solutions. To see that there can be no other solutions, you can convince yourself that for any choice of initial values $y(a) = b_0$, $y'(a) = b_1$ there exist values of A and B such that $y = A e^{2x} + B e^{-x}$ takes these initial values. By uniqueness, this means there can be no other solutions. We will soon discuss this issue more closely.

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