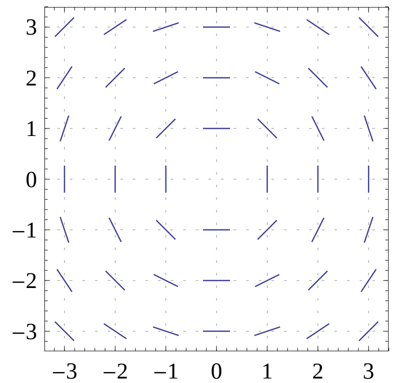
Understanding DEs without solving them

Slope fields, or sketching solutions

Example 9. Consider the DE $y' = -x/y$.

Let's pick a point, say, $(1,2)$. If a solution $y(x)$ is passing through that point, then its slope has to be $y' = -1/2$. We therefore draw a small line through the point $(1,2)$ with slope $-1/2$. Continuing in this fashion for several other points, we obtain the slope field on the right.



With just a little bit of imagination, we can now anticipate the solutions to look like (half)circles around the origin. Let us check whether $y(x) = \sqrt{r^2 - x^2}$ might indeed be a solution!

$y'(x) = \frac{1}{2}\frac{-2x}{\sqrt{r^2 - x^2}} = -x/y(x)$. So, yes, we actually found solutions! ◇

Existence and uniqueness of solutions

Definition 10. A solution to the IVP $y' = f(x,y)$, $y(a) = b$ is a function $y(x)$, defined on an interval I containing a, such that $y'(x) = f(x, y(x))$ for all $x \in I$ and $y(a) = b$.

Theorem 11. [1]Consider the IVP $y' = f(x,y)$, $y(a) = b$.

(i) If $f(x,y)$ is continuous [in a rectangle] around (a,b), then there exists a (local[2]) solution.

(ii) If both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous [in a rectangle] around (a,b), then there exists a (locally[3]) unique solution.

Example 12. Consider, again, the IVP $y' = -x/y$, $y(a) = b$.

Here, $f(x,y) = -x/y$ and $\frac{\partial}{\partial y}f(x,y) = x/y^2$. Both are continuous for all (x,y) with $y \neq 0$. Hence, if $b \neq 0$ then the IVP has a unique solution.
Assume $b > 0$ (things work similarly for $b < 0$). Then $y(x) = \sqrt{r^2 - x^2}$ solves the IVP if we choose $r = \sqrt{a^2 + b^2}$.
Uniqueness means that there is no other solution to the IVP than this one (which we had guessed). ◇

Example 13. Discuss the IVP $y' = y$, $y(a) = b$.

Solution. Here, $f(x,y) = y$ and both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous for all (x,y). That means the IVP always has a unique solution (at least locally).

As a consequence, there can be no other solutions to the DE $y' = y$ than the ones of the form $y(x) = Ce^x$. Why?! [Assume that $y(x)$ satisfies $y' = y$ and let (a,b) any value on the graph of y. Then $y(x)$ solves the IVP $y' = y$, $y(a) = b$; but so does Ce^x with $C = b/e^a$. The uniqueness implies that $y(x) = Ce^x$.] ◇

Example 14. Last time, we verified that $y' = y^2$, $y(0) = 1$ is solved by $y(x) = \frac{1}{1-x}$ on $(-\infty, 1)$.

Note that $y(x)$ cannot be continuously extended past $x = 1$; it is only a local solution! As in the previous example, we find that it is the unique solution to the IVP. ◇

1. The two parts of the theorem are famous results usually attributed to Peano and Picard–Lindelöf.
2. We call this a local solution at a, to emphasize that the interval I could be very small.
3. The interval in which the solution is unique could be smaller than the interval in which it exists.

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