Sketch of Lecture 2

Wed, 01/22/2014

Review. Verify that $x(t) = \frac{1}{c - kt}$ is a one-parameter family of solutions to $\frac{dx}{dt} = kx^2$.

Solution. $x'(t) = \frac{k}{(c-kt)^2} = kx(t)^2$

Solve the IVP: $\frac{dx}{dt} = kx^2$, $x(0) = 2$. [What about $x(0) = 0$, instead?]

Solution. $\left[\frac{1}{c-kt}\right]_{t=0} = \frac{1}{c} \overset{!}{=} 2$. Hence $c = \frac{1}{2}$. Solution $x(t) = \frac{1}{1/2 - kt}$. [Solution $x(t) = 0$. This may be seen as the case $c \to \infty$.] $\diamondsuit$

Why care about DEs?

A very simple model of population growth

If $P(t)$ is the size of a population (eg. of bacteria) at time t, then the rate of change $\frac{dP}{dt}$ might, from biological considerations, be (nearly) proportional to $P(t)$.

The corresponding mathematical model is described by the DE $P' = kP$ where k is the constant of proportionality.

Mathematics (which we will soon have learned) tells us that the (only) solutions to this DE are given by $P(t) = Ce^{kt}$ where C is some constant. (Hence, populations satisfying the assumption from biology necessarily exhibit exponential growth.)

Example 6. Suppose $P(0) = 100$ and $P(1) = 300$. Find $P(t)$.

Solution. $Ce^{k\cdot 0} = C = 100$ and $Ce^k = 100e^k = 300$. Hence $k = \ln(3)$ and $P(t) = 100 e^{\ln(3)\,t} = 100 \cdot 3^t$. $\diamondsuit$

Main problem of modeling: a model has to be detailed enough to resemble the real world, yet simple enough to allow for mathematical analysis.

The logistic model of population growth

If the population is constrained by resources, then $P' = kP$ is not a good model. A model to take that into account is $\frac{dP}{dt} = kP(1 - P/M)$. This is the logistic equation (M is called the carrying capacity). Note that if $P \ll M$ then $1 - P/M \approx 1$ and we are back to the simpler model.

On the other hand, if $P > M$ then $1 - P/M < 0$ so that (assuming $k > 0$) $P' < 0$ (i.e. population is shrinking).

Example 7. We will learn to solve such DEs. For now, we can still verify that

$$P(t) = \frac{CMe^{kt}}{M + C(e^{kt} - 1)}$$

solves the logistic DE. Note that $P(0) = C$. Also, $\lim_{t\to\infty} P(t) = M$ (using, for instance, L'Hospital). $\diamondsuit$

Movement of objects, velocity and acceleration

Example 8. A ball is dropped from a 100m tall building. How long until it reaches the ground? What's the speed when it hits the ground?

Solution. Let $x(t)$ be the height at which the ball is at time t. Velocity is $v(t) = x'(t)$ and acceleration $a(t) = x''(t)$. For a falling object, $a(t) = -g$ (on earth, the graviational acceleration is $g \approx 9.8\text{m/s}^2$) and hence $x(t)$ solves the DE $x'' = -g$. We also know the initial values $x(0) = 100$, $x'(0) = 0$.

Hence, $x'(t) = -gt$ and, therefore, $x(t) = -\frac{1}{2}gt^2 + 100$.

It reaches the ground when $x(t) = -\frac{1}{2}gt^2 + 100 = 0$, that is after $t = \sqrt{200/g} \approx 4.5$ seconds.

The speed is $|x'(4.5)| \approx 44.1\,\text{m}/s$. $\diamondsuit$