Preparing the Final

MATH 286 — Differential Equations Plus
May 9, 2014

Read Euler, read Euler, he is the master of us all.
— Pierre-Simon Laplace (1749–1827) —

Problem 1.

(a) Find the Fourier series of the function of period 2 characterized by

$$f(t) = \begin{cases} t, & \text{for } 0 \leqslant t < 1, \\ t+2, & \text{for } 1 \leqslant t < 2. \end{cases}$$

(b) Let $g(t)$ be the sum of the Fourier series you just calculated. Sketch the graph of $g(t)$. What are $g(0)$, $g(1)$ and $g(2)$? Explain the general phenomenon.

Problem 2. A mass-spring system is described by the equation

$$mx'' + x = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} \sin\left(\frac{nt}{3}\right).$$

(a) For which m does pure resonance occur?

(b) Find the general solution when $m = 1/9$.

Problem 3. Let $f(t) = 1$ for $t \in (0, L)$.

(a) Extend $f(t)$ to an odd $2L$-periodic function $f_o(t)$. Sketch the graph of the sum of the Fourier series of $f_o(t)$.

(b) Calculate the Fourier series of $f_o(t)$ with period $2L$. (This is also known as the Fourier sine series of $f(t)$.)

(c) Explain, using the heat equation as an example, why it can be useful to write a *constant function* as an infinite sum of sine terms.

Problem 4. For which values of λ does the boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0$$

have nonzero solutions? Find all these solutions. Make sure to consider all cases.

Problem 5. Find the solution $u(x,t)$, for $0 < x < 3$ and $t \geqslant 0$, to the heat conduction problem

$$2u_t = u_{xx}, \quad u_x(0,t) = 0, \; u(3,t) = 0, \quad u(x,0) = 2\cos\left(\frac{\pi x}{2}\right) + 7\cos\left(\frac{3\pi x}{2}\right).$$

Derive your solution using separation of variables (at some step you may refer to the previous problem). Don't rely on a formula.

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Problem 6. Using the Laplace transform, solve the initial value problem $x'' + 4x' + 4x = f(t)$ with $x(0) = 0$, $x'(0) = 0$ and

$$f(t) = \begin{cases} 2, & \text{for } 0 \leqslant t < 2, \\ t, & \text{for } 2 \leqslant t < 3, \\ 1, & \text{for } t \geqslant 3. \end{cases}$$

Finally, here is the table for the Laplace transform, which you will be given for the final exam.

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$tf(t)$	$-F'(s)$
$u_a(t)f(t-a)$	$e^{-sa}F(s)$

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