Midterm #2

MATH 238 — Differential Equations I
Wednesday, Nov 12, 2025

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 36 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (4 points) Assume that the angle $\theta(t)$ of a swinging pendulum is described by $\theta'' + 4\theta = 0$. Suppose $\theta(0) = 3$, $\theta'(0) = 8$. What are the period and the amplitude of the resulting oscillations?

Solution. The characteristic equation has roots $\pm 2i$. Hence, the general solution to the DE is $\theta(t) = A\cos(2t) + B\sin(2t)$.

We use the initial conditions to determine A and B: $\theta(0) = A \overset{!}{=} 3$. $\theta'(0) = 2B \overset{!}{=} 8$.

Hence, the unique solution to the IVP is $\theta(t) = 3\cos(2t) + 4\sin(2t)$.

In particular, the period is $2\pi/2 = \pi$ and the amplitude is $\sqrt{A^2 + B^2} = \sqrt{3^2 + 4^2} = 5$.

Problem 2. (4 points) Consider the following system of initial value problems:

$$\begin{aligned} y_1'' - 3y_1 &= y_2 + 8 \\ y_2'' + 4y_2 &= 2y_1 - 7y_1' \end{aligned} \qquad y_1(0) = 2,\ y_1'(0) = 0,\ y_2(0) = 1,\ y_2'(0) = 6$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y} + \boldsymbol{f}$, $\boldsymbol{y}(0) = \boldsymbol{c}$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 \\ 2 & -4 & -7 & 0 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 0 \\ 0 \\ 8 \\ 0 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 6 \end{bmatrix}.$$

Problem 3. (3 points) The position $y(t)$ of a certain mass on a spring is described by $2y'' + ky = 5\cos(t) - \cos(3t)$. For which values of k, if any, does resonance occur?

Solution. The natural frequency is $\sqrt{\frac{k}{2}}$ while the external frequencies are 1 and 3. Resonance therefore occurs if $\sqrt{\frac{k}{2}} = 1$ or $\sqrt{\frac{k}{2}} = 3$. Equivalently, if $k = 2$ or $k = 18$.

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Problem 4. (10 points) Determine the general solution of the following system: $\begin{array}{rcl} y_1' & = & y_1 + 4y_2 - 6e^{3x} \\ y_2' & = & y_1 - 2y_2 \end{array}$

Solution. Using $y_1 = y_2' + 2y_2$ (from the second equation) in the first equation, we get $y_2'' + 2y_2' = (y_2' + 2y_2) + 4y_2 + 6e^{3x}$.

Simplified, this is $y_2'' + y_2' - 6y_2 = -6e^{3x}$. This is an inhomogeneous linear DE with constant coefficients. Note that $D^2 + D - 6 = (D-2)(D+3)$. Since the characteristic roots for the homogeneous DE are $2, -3$, while the root for the inhomogeneous part is 3, there must a particular solution of the form $y_2 = Ae^{3x}$. Plugging this y_2 into the DE, we get $y_2'' + y_2' - 6y_2 = (3^2 + 3 - 6)Ae^{3x} = 6Ae^{3x} \overset{!}{=} -6e^{3x}$. Hence, $A = -1$ and the particular solution is $y_2 = -e^{3x}$. The corresponding general solution is $y_2 = -e^{3x} + C_1e^{2x} + C_2e^{-3x}$.

Correspondingly, $y_1 = y_2' + 2y_2 = (-3e^{3x} + 2C_1e^{2x} - 3C_2e^{-3x}) + 2(-e^{3x} + C_1e^{2x} + C_2e^{-3x}) = -5e^{3x} + 4C_1e^{2x} - C_2e^{-3x}$.

Problem 5. (2 points) The motion of a certain mass on a spring is described by $my'' + y' + 3y = 0$ where $m > 0$. For which values of m is the motion overdamped?

Solution. The discriminant of the characteristic equation is $1 - 12m$. Hence the system is overdamped if $1 - 12m > 0$, that is $m < \frac{1}{12}$.

Problem 6. (6 points) The mixtures in two tanks T_1, T_2 are kept uniform by stirring. Brine containing 2 lb of salt per gallon enters T_1 at 5 gal/min, and the solution is pumped at a rate of 4 gal/min into T_2. Finally, solution is leaving T_2 at 4 gal/min. Initially, T_1 and T_2 contain 40gal of pure water each.

Denote by $y_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the y_i, including initial conditions. (Do *not* attempt to solve the system.)

Solution. Note that at time t, T_1 contains $40 + t$ gal of solution while T_2 contains a constant amount of 40 gal.

In the time interval $[t, t + \Delta t]$, we have:

$$\begin{array}{rcl} \Delta y_1 \approx 5 \cdot 2 \cdot \Delta t - 4 \cdot \dfrac{y_1}{40+t} \cdot \Delta t & \Longrightarrow & y_1' = 10 - \dfrac{4}{40+t} y_1 \\ \Delta y_2 \approx 4 \cdot \dfrac{y_1}{40+t} \cdot \Delta t - 4 \cdot \dfrac{y_2}{40} \cdot \Delta t & \Longrightarrow & y_2' = \dfrac{4}{40+t} y_1 - \dfrac{1}{10} y_2 \end{array}$$

We also have the initial conditions $y_1(0) = 0,\ y_2(0) = 0$.

Optional: in matrix form, writing $\boldsymbol{y} = (y_1, y_2)$, this takes the form

$$\boldsymbol{y}' = \begin{bmatrix} -\frac{4}{40+t} & 0 \\ \frac{4}{40+t} & -\frac{1}{10} \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Problem 7. (7 points) Fill in the blanks. None of the problems should require any computation!

(a) Let y_p be any solution to the inhomogeneous linear differential equation $y'' + 5y = 4 - 2xe^{3x}$. Find a homogeneous linear differential equation which y_p solves.

Here, and in the next part, you can use the operator D to write the DE. No need to simplify, any form is acceptable.

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(b) Write down a homogeneous linear differential equation satisfied by $y(x) = (2 + 3x)\, e^{-x} - 3x^2$.

(c) Consider a homogeneous linear differential equation with constant real coefficients which has order 4. Suppose $y(x) = 5e^{2x}\cos(3x) + 7x\, e^{x}$ is a solution. Write down the general solution.

(d) Name the method which we can use to solve the differential equation $y'' + y' - 6y = 4\ln(x)$.

Solution.

(a) $D(D-3)^2(D^2+5)y = 0$

Explanation. Since y_p solves the inhomogeneous DE, we have $(D^2+5)y_p = 4 - 2x e^{3x}$. The inhomogeneous part is a solution of $p(D)y = 0$ if and only if $0, 3, 3$ are roots of the characteristic polynomial $p(D)$. In particular, $D(D-3)^2(4 - 2x e^{3x}) = 0$. Combined, we find that $D(D-3)^2(D^2+5)y_p = 0$.

(b) $(D+1)^2 D^3 y = 0$

Explanation. In order for $y(x)$ to be a solution of $p(D)y = 0$, we need to have the characteristic roots -1, $-1, 0, 0, 0$. Hence, the simplest DE is $(D+1)^2 D^3 y = 0$.

(c) $y(x) = C_1 e^{2x}\cos(3x) + C_2 e^{2x}\sin(3x) + (C_3 + C_4 x)e^{x}$.

Explanation. $y(x) = 5e^{2x}\cos(3x) + 7x\, e^{x}$ is a solution of $p(D)y = 0$ if and only if $2 \pm 3i, 1, 1$ are roots of the characteristic polynomial $p(D)$. Since the order of the DE is 4, there can be no further roots. Hence, the general solution of this DE is $y(x) = C_1 e^{2x}\cos(3x) + C_2 e^{2x}\sin(3x) + (C_3 + C_4 x)e^{x}$.

(d) Variation of constants

Note that we cannot use the method of undetermined coefficients here because the inhomogeneous part $4\ln(x)$ is not a solution of a DE of the form $p(D)y = 0$.

(extra scratch paper)