Notes for Lecture 1

Wed, 8/20/2025

Review: Computing derivatives

Given a function $y(x)$, we learned in Calculus I that its **derivative**

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(where $\Delta y = y(x + \Delta x) - y(x)$) has the following two important characterizations:

- $y'(x)$ is the **slope of the tangent line** of the graph of $y(x)$ at x, and
- $y'(x)$ is the **rate of change** of $y(x)$ at x.

Comment. Derivatives were introduced in the late 1600s by Newton and Leibniz who later each claimed priority in laying the foundations for calculus. Certainly both of them contributed mightily to those foundations.

Moreover, we learned simple rules to compute the derivative of functions:

- **(sum rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = f'(x) + g'(x)$
- **(product rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- **(chain rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x)$

  Comment. If we write $t = g(x)$ and $y = f(t)$, then the chain rule takes the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}$.
  In other words, the chain rule expresses the fact that we can treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ (which initially is just a notation for $y'(x)$) as an honest fraction.

- **(basic functions)** $\frac{\mathrm{d}}{\mathrm{d}x} x^r = r x^{r-1}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$,
  $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$

These rules are enough to compute the derivative of any function that we can build from the basic functions using algebraic operations and composition. On the other hand, as you probably recall from Calculus II, reversing the operation of differentiation (i.e. computing antiderivatives) is much more difficult.

In particular, there exist simple functions (such as e^{x^2}) whose antiderivative cannot be expressed in terms of the basic functions above.

Example 1. Derive the **quotient rule** from the rules above.

Solution. We write $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ and apply the product rule to get

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x)\frac{1}{g(x)} + f(x)\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)}.$$

By the chain rule combined with $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x} = -\frac{1}{x^2}$, we have $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)} = -\frac{1}{g(x)^2} g'(x)$. Using this in the previous formula,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x)\frac{1}{g(x)} - f(x)\frac{1}{g(x)^2} g'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2}.$$

Putting the final two fractions on a common denominator, we obtain the familiar quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Armin Straub
straub@southalabama.edu

Example 2. Compute the following derivatives:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(5x^3+7x^2+2)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(5x^3+7x^2+2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x)\sin(5x^3+7x^2+2)$

Solution.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(5x^3+7x^2+2)=15x^2+14x$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(5x^3+7x^2+2)=(15x^2+14x)\cos(5x^3+7x^2+2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x)\sin(5x^3+7x^2+2)$
$=(3x^2+2)\sin(5x^3+7x^2+2)+(x^3+2x)(15x^2+14x)\cos(5x^3+7x^2+2)$

First examples of differential equations

Example 3. Here are two first examples of a **differential equation** (DE):

(a) $y'=2xy$

This is short for $y'(x)=2xy(x)$. The goal is to find a function $y(x)$ satisfying this equation.

One such **solution** is $y(x)=e^{x^2}$. We will soon learn techniques to find this ourselves but, already now, we can verify that it is indeed a solution: if $y(x)=e^{x^2}$ then $y'(x)=2xe^{x^2}=2xy(x)$.

(b) $(xy'-4y''')^2=5\sin(2x+y^4)+7$

This illustrates that y and its derivative can show up in any kind of way. We say that this DE has **order** 3 because the highest derivative is the 3rd derivative y'''.

Example 4. Solve the DE $y'=x^2+x$.

Solution. Note that the DE simply asks for a function $y(x)$ with a specific derivative (in particular, the right-hand side does not involve $y(x)$). In other words, the desired $y(x)$ is an **antiderivative** of x^2+x. We know from Calculus II that we can find antiderivatives by integrating:

$$y(x)=\int(x^2+x)\mathrm{d}x=\frac{1}{3}x^3+\frac{1}{2}x^2+C$$

Moreover, we know from Calculus II that there are no other solutions. In other words, we found the **general solution** to the DE.

Example 5. As in the previous example, any DE of the form $y'=f(x)$ (this is artificially easy) is just asking us to compute an antiderivative of $f(x)$.

On the other hand, this is an early indication that solving DEs is hard (and includes computing integrals as a special case). For instance, the DE $y'=e^{x^2}$ requires us to compute the antiderivative of e^{x^2}. It turns out that this cannot be done using the basic functions we know from Calculus.

Advanced comment. A "solution" to the above issue is to **define** a new function as the antiderivative that we presently cannot write down a formula for. Look up the so-called **error function** if you are curious!

Example 6. (review) Solve the DE $y' = x^2 + x$.

Solution. Note that the DE simply asks for a function $y(x)$ with a specific derivative (in particular, the right-hand side does not involve $y(x)$). In other words, the desired $y(x)$ is an **antiderivative** of $x^2 + x$. We know from Calculus II that we can find antiderivatives by integrating:

$$y(x) = \int (x^2 + x)\mathrm{d}x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Moreover, we know from Calculus II that there are no other solutions. In other words, we found the **general solution** to the DE.

If the highest derivative appearing in a DE is an rth derivative, we say that the DE has **order** r.

For instance. The DE $y' = 3\sqrt{1 - y^2}$ has order 1 (such DEs are also called first order DEs).
On the other hand, the DE $y'' = y' + 6y$ has order 2 (such DEs are also called second order DEs).

As we will observe in the next few examples, we typically expect that the general solution of a DE of **order** r has r **parameters** (or degrees of freedom).

A first initial value problem

To single out a **particular solution**, we need to specify additional conditions (typically one condition per parameter in the general solution). For instance, it is common to impose **initial conditions** such as $y(1) = 2$. A DE together with an initial condition is called an **initial value problem** (IVP).

Example 7. Solve the IVP $y' = x^2 + x$ with $y(1) = 2$.

Solution. From the previous example, we know that $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$.
Since $y(1) = \frac{1}{3} + \frac{1}{2} + C = \frac{5}{6} + C \overset{!}{=} 2$, we find $C = 2 - \frac{5}{6} = \frac{7}{6}$.
Hence, $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{7}{6}$ is the (unique) solution of the IVP.

Example 8. (homework) Solve the DE $y'' = x^2 + x$.

Solution. We now take two antiderivatives of $x^2 + x$ to get

$$y(x) = \int\int (x^2 + x)\mathrm{d}x\mathrm{d}x = \int \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + C\right)\mathrm{d}x = \frac{1}{12}x^4 + \frac{1}{6}x^3 + Cx + D,$$

where it is important that we give the second constant of integration a name different from the first.

Important comment. This is the general solution to the DE. The DE is of order 2 and, as expected, the general solution has 2 parameters.

Verifying if a function solves a DE

Given a function, we can always check whether it solves a DE!

We can just plug it into the DE and see if left and right side agree. This means that we can always check our work as well as that we can verify solutions generated by someone else (or a computer algebra system) even if we don't know the techniques for solving the DE.

Armin Straub
straub@southalabama.edu

Example 9. (warmup) Consider the DE $y'' = y' + 6y$.

(a) Is $y(x) = e^{2x}$ a solution?

(b) Is $y(x) = e^{3x}$ a solution?

Solution.

(a) Starting with $y(x) = e^{2x}$, we compute $y' = 2e^{2x}$ and $y'' = 4e^{2x}$.
Since $y' + 6y = 8e^{2x}$ is different from $y'' = 4e^{2x}$, we conclude that $y(x) = e^{2x}$ is not a solution.

(b) Starting with $y(x) = e^{3x}$, we compute $y' = 3e^{3x}$ and $y'' = 9e^{3x}$.
Since $y' + 6y = 9e^{3x}$ is equal to $y'' = 9e^{3x}$, we conclude that $y(x) = e^{3x}$ is a solution of the DE.

We will soon be able to completely solve differential equations such as in the previous example. The following gives a taste of how we will go about it:

Example 10. (cont'd) Consider the DE $y'' = y' + 6y$. For which r is e^{rx} a solution?

Solution. If $y(x) = e^{rx}$, then $y'(x) = r e^{rx}$ and $y''(x) = r^2 e^{rx}$.
Plugging $y(x) = e^{rx}$ into the DE, we get $r^2 e^{rx} = r e^{rx} + 6e^{rx}$ which simplifies to $r^2 = r + 6$.
This has the two solutions $r = -2$, $r = 3$. Hence e^{-2x} and e^{3x} are solutions of the DE.

In fact, we can check that $Ae^{-2x} + Be^{3x}$ is a **two-parameter family** of solutions to the DE.

Important comment. It is no coincidence that the order of the DE is 2, whereas the previous example has order 1. In general, we expect a DE of order r to have a solution with r parameters.

Example 11. Consider the DE $e^y y' = 1$.

(a) Is $y(x) = \ln(x)$ a solution to the DE?

(b) Is $y(x) = \ln(x) + C$ a solution to the DE?

(c) Is $y(x) = \ln(x + C)$ a solution to the DE?

Solution.

(a) Since $y'(x) = \frac{1}{x}$ and $e^{y(x)} = e^{\ln(x)} = x$, we have $e^y y' = x \cdot \frac{1}{x} \stackrel{\checkmark}{=} 1$.
Hence, $y(x) = \ln(x)$ is a solution to the given DE.

(b) Since $y'(x) = \frac{1}{x}$ and $e^{y(x)} = e^{\ln(x) + C} = x e^C$, we have $e^y y' = x e^C \cdot \frac{1}{x} = e^C$. Thus the DE is satisfied only if $e^C = 1$ which only happens if $C = 0$ (which is the case in the first part).
Hence, $y(x) = \ln(x) + C$ is not a solution to the given DE except if $C = 0$.

(c) Since $y'(x) = \frac{1}{x + C}$ and $e^{y(x)} = e^{\ln(x + C)} = x + C$, we have $e^y y' = (x + C) \cdot \frac{1}{x + C} \stackrel{\checkmark}{=} 1$.
Hence, $y(x) = \ln(x + C)$ is indeed a one-parameter family of solutions to the given DE.

Armin Straub
straub@southalabama.edu

Any fixed function solves many DEs

Usually, we start with a DE (which comes, for instance, from physical laws) and want to solve it. In the next example, we start with a function and determine several DEs that it solves.

Example 12. Determine several (random) DEs that $y(x) = \sin(3x)$ solves.

Solution. Here are some options (but there are many more):

(a) We compute $y'(x) = 3\cos(3x)$. Accordingly, $y(x) = \sin(3x)$ solves the DE $y' = 3\cos(3x)$.

Comment. This, however, is not an "interesting" choice. In particular, this DE could be simply solved by computing an antiderivative (as in the previous examples).

Comment. Note that there are further solutions to this DE: the **general solution** is $\int 3\cos(3x)\mathrm{d}x = \sin(3x) + C$ where C is any constant. We say that $y(x) = \sin(3x) + C$ is a **one-parameter family** of solutions to the DE. C is called a **degree of freedom**.

(b) Note that $y'(x) = 3\cos(3x) = 3\sqrt{1 - (\sin(3x))^2} = 3\sqrt{1 - y(x)^2}$ (for x close to 0).

[Here we used that $\cos(x)^2 + \sin(x)^2 = 1$, which implies that $\cos(x) = \sqrt{1 - \sin(x)^2}$.]

Hence, $y(x) = \sin(3x)$ solves the differential equation $y' = 3\sqrt{1 - y^2}$.

Comment. In the above, we restrict x to $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$ so that $\cos(3x) > 0$. Less precisely, we can say that x is close to 0. (It is a common feature of DEs that we work with values of x close to a certain initial value.)

(c) We compute $y''(x) = -9\sin(3x)$. Accordingly, $y(x) = \sin(3x)$ solves the DE $y'' = -9\sin(3x)$.

Comment. Once more this DE is easy (because it only involves y'' but not y or y'). Hence, we can find the general solution by simply taking two antiderivatives:

$$y(x) = \iint -9\sin(3x)\mathrm{d}x\,\mathrm{d}x = \int (3\cos(3x) + C)\mathrm{d}x = \sin(3x) + Cx + D.$$

It is important that we give the second constant of integration a name different from the first. That way, we see that the general solution has 2 degrees of freedom. This matches the fact that the order of the DE is 2.

Important comment. This is no coincidence. In general, we expect a DE of order r to have a general solution with r parameters.

(d) $y(x) = \sin(3x)$ also solves the DE $y'' = -9y$.

Comment. This is again a DE of order 2. Therefore the general solution should have 2 degrees of freedom. Later we will learn to solve such DEs. For now, we can verify that $y(x) = A\sin(3x) + B\cos(3x)$ is a solution for any constants A and B.

Homework. Check that $y(x) = \sin(3x) + C$ does not solve the DE $y'' = -9y$.

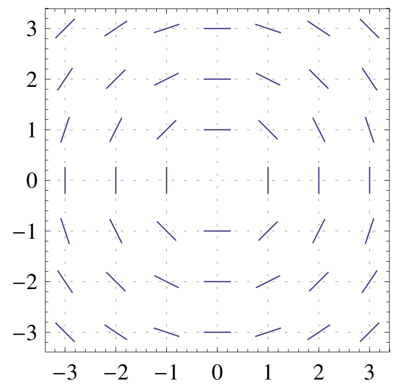
Slope fields, or sketching solutions to DEs

The next example illustrates that we can "plot" solutions to differential equations (it does not matter if we are able to actually solve them).

Comment. This is an important point because "plotting" really means that we can numerically approximate solutions. For complicated systems of differential equations, such as those used to model fluid flow, this is usually the best we can do. Nobody can actually solve these equations.

Example 13. Consider the DE $y' = -x/y$.

Let's pick a point, say, $(1,2)$. If a solution $y(x)$ is passing through that point, then its slope has to be $y' = -1/2$. We therefore draw a small line through the point $(1,2)$ with slope $-1/2$. Continuing in this fashion for several other points, we obtain the **slope field** on the right.



With just a little bit of imagination, we can now anticipate the solutions to look like (half)circles around the origin. Let us check whether $y(x) = \sqrt{r^2 - x^2}$ might indeed be a solution!

$y'(x) = \frac{1}{2}\frac{-2x}{\sqrt{r^2 - x^2}} = -x/y(x)$. So, yes, we actually found solutions!

Solving DEs: Separation of variables

Example 14. Solve the DE $y' = -\frac{x}{y}$.

Solution. Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

Separate the variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$ (in particular, we are multiplying both sides by $\mathrm{d}x$).

Integrating both sides, we get $\int y\,\mathrm{d}y = \int -x\,\mathrm{d}x$.

Computing both integrals results in $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ (we combine the two constants of integration into one).

Hence $x^2 + y^2 = D$ (with $D = 2C$).

This is an **implicit form** of the solutions to the DE. We can make it explicit by solving for y. Doing so, we find $y(x) = \pm\sqrt{D - x^2}$ (choosing $+$ gives us the upper half of a circle, while the negative sign gives us the lower half).

Comment. The step above where we break $\frac{\mathrm{d}y}{\mathrm{d}x}$ apart and then integrate may sound sketchy!

However, keep in mind that, after we find a solution $y(x)$, even if by sketchy means, we can (and should!) verify that $y(x)$ is indeed a solution by plugging into the DE. We actually already did that in the previous example!

Example 15. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = -3$.

Comment. Instead of using what we found in Example 14, we start from scratch to better illustrate the solution process (and how to use the initial condition right away to determine the value of the constant of integration).

Solution. We separate variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$.

Integrating gives $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, and we use $y(0) = -3$ to find $\frac{1}{2}(-3)^2 = 0 + C$ so that $C = \frac{9}{2}$.

Hence, $x^2 + y^2 = 9$ is an **implicit** form of the solution.

Solving for y, we get $y = -\sqrt{9 - x^2}$ (note that we have to choose the negative sign so that $y(0) = -3$).

Comment. Note that our solution is a **local solution**, meaning that it is valid (and solves the DE) locally around $x = 0$ (from the initial condition). However, it is not a **global solution** because it doesn't make sense outside of x in the interval $[-3, 3]$.

Armin Straub
straub@southalabama.edu

Which differential equations can we actually solve using separation of variables?

- A general DE of first order is typically of the form $\frac{dy}{dx} = f(x, y)$.

  For instance, $\frac{dy}{dx} = \sin(xy) - x^2 y$.

  Comment. First order means that only the first derivative of y shows up. The most general form of a DE of first order is $F(x, y, y') = 0$ but we can usually solve for y' to get to the above form.

- The ones we can solve are **separable equations**, which are of the form $\frac{dy}{dx} = g(x)h(y)$.

  Example. The equation $\frac{dy}{dx} = y - x$ (although simple) is not separable.

  Example. The equation $\frac{dy}{dx} = e^{y-x}$ is separable because we can write it as $\frac{dy}{dx} = e^y e^{-x}$.

Example 16. (extra)

Comment. In this example, we use $x(t)$ instead of $y(x)$ for the function described by the differential equation. In general, of course, any choice of variable names is possible. If we write something like x' or y' it needs to be clear from the context with respect to which variable that derivative is meant (such as $x' = \frac{d}{dt}x(t)$).

(a) Solve the DE $\frac{dx}{dt} = kx^2$.

(b) Verify your answer from the first part.

(c) Solve the IVP $\frac{dx}{dt} = kx^2$, $x(0) = 2$.

(d) Solve the IVP $\frac{dx}{dt} = kx^2$, $x(0) = 0$.

Solution.

(a) This DE is separable: $\frac{1}{x^2}dx = k\,dt$. Integrating, we find $-\frac{1}{x} = kt + B$. (We plan to replace B by a new constant C in a moment.) Hence, $x = -\frac{1}{kt + B} = \frac{1}{C - kt}$.

[Here, $C = -B$ but that relationship doesn't matter; it only matters that the solution has a free parameter.]

Comment. Note that we did not find the solution $x = 0$ (lost when dividing by x^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). [Although, we can obtain it from the general solution by letting $C \to \infty$.]

See the last part for a case when this "missing" solution is needed.

(b) Starting with $x(t) = \frac{1}{C - kt}$, we compute that $\frac{dx}{dt} = -\frac{1}{(C - kt)^2} \cdot (-k) = \frac{k}{(C - kt)^2}$.

On the other hand, $kx^2 = k\left(\frac{1}{C - kt}\right)^2 = \frac{k}{(C - kt)^2}$. Since this matches what we got for $\frac{dx}{dt}$, it is indeed true that $\frac{dx}{dt} = kx^2$.

(c) We start with $x(t) = \frac{1}{C - kt}$ (which we know solves the DE for any value of C) and seek to choose C so that $x(0) = 2$.

Since $x(0) = \left[\frac{1}{C - kt}\right]_{t=0} = \frac{1}{C} \stackrel{!}{=} 2$, we find $C = \frac{1}{2}$.

Hence, the IVP has the (unique) solution $x(t) = \frac{1}{1/2 - kt}$.

(d) Proceeding as in the previous part, we now arrive at the impossible equation $\frac{1}{C} \stackrel{!}{=} 0$.

However, this suggests that we should consider taking $C \to \infty$ in $x(t) = \frac{1}{C - kt}$, which results in $x(t) = 0$.

Indeed, it is easy to verify (make sure you know what this entails!) that $x(t) = 0$ solves the IVP.

Armin Straub
straub@southalabama.edu

In the following example, we first proceed like we did when producing a slope field to compute slopes (and, therefore, tangent lines) of solutions. Indeed, besides the slope y', we can compute further derivatives like y'' or y''' by differentiating the DE.

Do you recall how y'' tells us about the curvature of a function $y(x)$?

Example 17. Consider the DE $x^2y' = 1 + xy^3$. Suppose that $y(x)$ is a solution passing through the point $(2, 1)$.

Important. This is the same as saying that $y(x)$ solves the IVP $x^2y' = 1 + xy^3$ with $y(2) = 1$.

(a) Determine $y'(2)$.

(b) Determine the tangent line of $y(x)$ at $(2, 1)$.

(c) Determine $y''(2)$.

Comment. Note that this DE is not separable.

Solution.

(a) At the point $(2, 1)$ we have $x = 2$ and $y = 1$. Plugging these values into the differential equation, we get $4y' = 1 + 2 \cdot 1^3 = 3$ which we can solve for y' to find $y' = \frac{3}{4}$.

Since y' is short for $y'(x) = y'(2)$, we have found $y'(2) = \frac{3}{4}$.

(b) The tangent line is the line through $(2, 1)$ with slope $\frac{3}{4}$ (computed in the previous part).

From this information, we can immediately write down its equation in the form $y = \frac{3}{4}(x - 2) + 1$.

(c) To get our hands on $y''(2)$, we can differentiate (with respect to x) both sides of $x^2y' = 1 + xy^3$.
Applying the product rule, we have $\frac{\mathrm{d}}{\mathrm{d}x}x^2y'(x) = 2xy'(x) + x^2y''(x) = 2xy' + x^2y''$ as well as $\frac{\mathrm{d}}{\mathrm{d}x}(1 + xy(x)^3) = y(x)^3 + x \cdot 3y(x)^2 \cdot y'(x) = y^3 + 3xy^2y'$.
Thus $2xy' + x^2y'' = y^3 + 3xy^2y'$. To find $y''(2)$, we plug in $x = 2$, $y = 1$, $y' = \frac{3}{4}$.

This results in $2 \cdot 2 \cdot \frac{3}{4} + 4y'' = 1 + 3 \cdot 2 \cdot 1 \cdot \frac{3}{4}$ or $3 + 4y'' = \frac{11}{2}$. It follows that $y'' = \frac{1}{4} \cdot \frac{5}{2} = \frac{5}{8}$.

Comment. Alternatively, we can rewrite the DE as $y' = \frac{1}{x^2} + \frac{1}{x}y^3$ and then differentiate. Do it!

Comment. Do you recall from Calculus what it means visually to have $y'' = \frac{5}{8}$?

[Since $y'' > 0$ it means that our function is concave up at $(2, 1)$. As such, its graph will lie above the tangent line.]

Comment. Note that we could continue and likewise find $y'''(2)$ or higher derivatives at $x = 2$. This is the starting point for the power series method typically discussed in Differential Equations II.

Example 18. (warm-up) Consider the DE $xy' = 2y$.

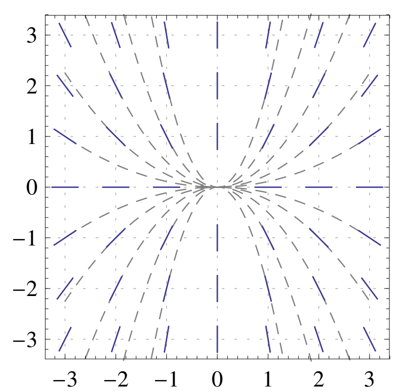
Sketch its slope field.

Challenge. Try to guess solutions $y(x)$ from the slope field.

Solution. For instance, to find the slope at the point $(3, 1)$, we plug $x = 3$, $y = 1$ into the DE to get $3y' = 2$. Hence, the slope is $y' = 2/3$.

The resulting slope field is sketched on the right.

Solution of the challenge. Trace out the solution through $(1, 1)$ (and then some other points). Their shape looks like a parabola, so that we might guess that $y(x) = Cx^2$ solves the DE. Check that this is indeed the case by plugging into the DE!



Armin Straub
straub@southalabama.edu

In general, **separation of variables** solves $y' = g(x)h(y)$ by writing the DE as $\frac{1}{h(y)}\mathrm{d}y = g(x)\mathrm{d}x$.

Note that $\frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$ is indeed equivalent to $\int \frac{1}{h(y)}\mathrm{d}y = \int g(x)\mathrm{d}x + C$. Why?! (Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to the integrals...)

Example 19. (cont'd) Solve the IVP $xy' = 2y$, $y(1) = 2$.

Solution. Rewrite the DE as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$.

Then multiply both sides with $\mathrm{d}x$ and integrate both of them to get $\int \frac{1}{y}\mathrm{d}y = \int \frac{2}{x}\mathrm{d}x$.

Hence, $\ln|y| = 2\ln|x| + C$.

The initial condition $y(1) = 2$ tells us that, at least locally, $x > 0$ and $y > 0$. Thus $\ln(y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = 2$, we find $C = \ln(2)$.

Solving $\ln(y) = 2\ln(x) + \ln(2)$ for y, we find $y = e^{2\ln(x) + \ln(2)} = 2x^2$.

Comment. When solving a DE or IVP, we can generally only expect to find a **local solution**, meaning that our solution might only be valid in a small interval around the initial condition (here, we can only expect $y(x)$ to be a solution for all x in an interval around 1; especially since we assumed $x > 0$ in our solution). However, we can check (do it!) that the solution $y = 2x^2$ is actually a **global solution** (meaning that it is a solution for all x, not just locally around 1).

Advanced comment. However, there are other global solutions such as $y(x) = \begin{cases} 2x^2, & \text{for } x \geqslant 0, \\ 7x^2, & \text{for } x < 0. \end{cases}$

Can you see that this piecewise defined function $y(x)$ is differentiable everywhere? (The slope at $x = 0$ is 0.)

The 7 can, of course, be replaced with any other number.

Let's solve the same differential equation with a different choice of initial condition:

Example 20. Solve the IVP $xy' = 2y$, $y(1) = -1$.

Solution. Again, we rewrite the DE as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$, multiply both sides with $\mathrm{d}x$, and integrate to get $\int \frac{1}{y}\mathrm{d}y = \int \frac{2}{x}\mathrm{d}x$.

Hence, $\ln|y| = 2\ln|x| + C$. The initial condition $y(1) = -1$ tells us that, at least locally, $x > 0$ and $y < 0$ (note that this means $|y| = -y$). Thus $\ln(-y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = -1$, we find $C = 0$.

Solving $\ln(-y) = 2\ln(x)$ for y, we find $y = -e^{2\ln(x)} = -x^2$. We easily verify that this is indeed a global solution.

Example 21. $y' = x + y$ is a DE for which the variables cannot be separated.

No worries, very soon we will have several tools to solve this DE as well.

Example 22. Solve $y' = 5y^4$ with $y(0) = 2$.

Solution. Separate variables to get $\frac{1}{y^4}\frac{\mathrm{d}y}{\mathrm{d}x} = 5$.

We integrate $\int \frac{1}{y^4}\mathrm{d}y = \int 5\mathrm{d}x$ to obtain the implicit solution $-\frac{1}{3y^3} = 5x + C$.

To find C, we plug in $x = 0$ and $y = 2$ (initial condition) to get $-\frac{1}{3 \cdot 2^3} = 5 \cdot 0 + C$ which simplifies to $C = -\frac{1}{24}$.

We now need to solve $-\frac{1}{3y^3} = 5x - \frac{1}{24}$ for y. We get $\frac{1}{y^3} = \frac{1}{8} - 15x$ so that $y^3 = \frac{1}{\frac{1}{8} - 15x}$.

Hence, the unique solution is $y(x) = \sqrt[3]{\frac{1}{\frac{1}{8} - 15x}}$.

Optional simplifications. Many people prefer to simplify this to $\sqrt[3]{\frac{8}{1 - 120x}}$ to avoid double fractions. This then further simplifies to $\frac{2}{\sqrt[3]{1 - 120x}}$.

Existence and uniqueness of solutions

The following is a very general result that allows us to guarantee that "nice" IVPs must have a solution and that this solution is unique.

Comment. Note that any first-order DE can be written as $g(y', y, x) = 0$ where g is some function of three variables. Assuming that g is reasonable, we can solve for y' and rewrite such a DE as $y' = f(x, y)$ (for some, possibly complicated, function f).

Comment. To be precise, a solution to the IVP $y' = f(x, y)$, $y(a) = b$ is a function $y(x)$, defined on an interval I containing a, such that $y'(x) = f(x, y(x))$ for all $x \in I$ and $y(a) = b$.

Theorem 23. (existence and uniqueness) Consider the IVP $y' = f(x, y)$, $y(a) = b$.
If both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous [in a rectangle] around (a, b), then the IVP has a unique solution in some interval $x \in (a - \delta, a + \delta)$ where $\delta > 0$.

Comment. The interval around a might be very small. In other words, the δ in the theorem could be very small.

Comment. Note that the theorem makes two important assertions. First, it says that there exists a **local solution**. Second, it says that this solution is unique. These two parts of the theorem are famous results usually attributed to Peano (existence) and Picard–Lindelöf (uniqueness).

Advanced comment. The condition about $\frac{\partial}{\partial y} f(x, y)$ is a bit technical (and not optimal). If we drop this condition, we still get existence but, in general, no longer uniqueness.

Advanced comment. The interval in which the solution is unique could be smaller than the interval in which it exists. In other words, it is possible that, away from the initial condition, the solution "forks" into two or more solutions. Note that this does not contradict the theorem because it only guarantees uniqueness on a small interval.

Example 24. Consider the IVP $(x - y^2)y' = 3x$, $y(4) = b$. For which choices of b does the existence and uniqueness theorem guarantee a unique (local) solution?

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = 3x/(x - y^2)$. We compute that $\frac{\partial}{\partial y} f(x, y) = 6xy/(x - y^2)^2$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $x - y^2 \neq 0$.

Note that $4 - b^2 \neq 0$ is equivalent to $b \neq \pm 2$.

Hence, if $b \neq \pm 2$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Armin Straub
straub@southalabama.edu

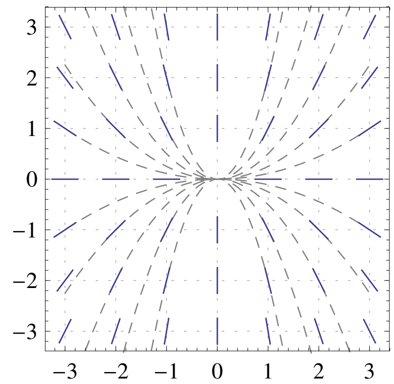
Example 25. Consider, again, the IVP $xy' = 2y$, $y(a) = b$. Discuss existence and uniqueness of solutions (without solving).

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = 2y/x$.

We compute that $\frac{\partial}{\partial y} f(x, y) = 2/x$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $x \neq 0$.

Hence, if $a \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.



What happens in the case $a = 0$?

Solution. In Example 18, we found that the DE $xy' = 2y$ is solved by $y(x) = Cx^2$.

This means that the IVP with $y(0) = 0$ has infinitely many solutions (see Example 19 for even more solutions).

On the other hand, the IVP with $y(0) = b$ where $b \neq 0$ has no solutions. (This follows from the fact that there are no solutions to the DE besides $y(x) = Cx^2$. Can you see this by looking at the slope field?)

Example 26. Consider the IVP $y' = ky^2$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = ky^2$. We compute that $\frac{\partial}{\partial y} f(x, y) = 2ky$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y).

Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Example 27. Solve $y' = ky^2$.

Solution. Separate variables to get $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = k$. Integrating $\int \frac{1}{y^2} \mathrm{d}y = \int k \mathrm{d}x$, we find $-\frac{1}{y} = kx + C$.

We solve for y to get $y = -\frac{1}{C + kx} = \frac{1}{D - kx}$ (with $D = -C$). That is the solution we verified earlier!

Comment. Note that we did not find the solution $y = 0$ (it was "lost" when we divided by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). However, note that we can obtain it from the general solution by letting $D \to \infty$.

Caution. We have to be careful about transforming our DE when using separation of variables: Just as the division by y^2 made us lose a solution, other transformations can add extra solutions which do not solve the original DE.

Here is a silly example (silly, because the transformation serves no purpose here) which still illustrates the point. The DE $(y-1)y' = (y-1)ky^2$ has the same solutions as $y' = ky^2$ plus the additional solution $y = 1$ (which does not solve $y' = ky^2$).

Example 28. Solve the IVP $y' = y^2$, $y(0) = 1$.

Solution. From the previous example with $k = 1$, we know that $y(x) = \frac{1}{D - x}$.

Using $y(0) = 1$, we find that $D = 1$ so that the unique solution to the IVP is $y(x) = \frac{1}{1 - x}$.

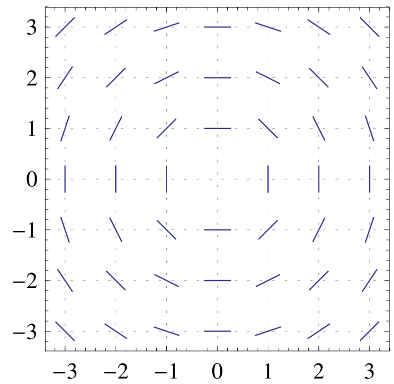
Comment. Note that we already concluded the uniqueness from the existence and uniqueness theorem.

On the other hand, note that $y(x) = \frac{1}{1 - x}$ is only valid on $(-\infty, 1)$ and that it cannot be continuously extended past $x = 1$; it is only a local solution.

Armin Straub
straub@southalabama.edu

Review. Existence and uniqueness theorem (Theorem 23) for an IVP $y' = f(x, y)$, $y(a) = b$:
If $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous around (a, b) then, locally, the IVP has a unique solution.

Example 29. Consider, again, the IVP $y' = -x/y$, $y(a) = b$. Discuss existence and uniqueness of solutions (without solving).



Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = -x/y$.

We compute that $\frac{\partial}{\partial y} f(x, y) = x/y^2$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $y \neq 0$.

Hence, if $b \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Comment. In Example 14, we found that the DE $y' = -x/y$ is solved by $y(x) = \pm\sqrt{D - x^2}$.
Assume $b > 0$ (things work similarly for $b < 0$). Then $y(x) = \sqrt{D - x^2}$ solves the IVP (we need to choose D so that $y(a) = b$) if we choose $D = a^2 + b^2$. This confirms that there exists a solution. On the other hand, uniqueness means that there can be no other solution to the IVP than this one.

What happens in the case $b = 0$?

Solution. In this case, the existence and uniqueness theorem does not guarantee anything. If $a \neq 0$, then $y(x) = \sqrt{a^2 - x^2}$ and $y(x) = -\sqrt{a^2 - x^2}$ both solve the IVP (so we certainly don't have uniqueness), however only in a weak sense: namely, both of these solutions are not valid locally around $x = a$ but only in an interval of which a is an endpoint (for instance, the IVP $y' = -x/y$, $y(2) = 0$ is solved by $y(x) = \pm\sqrt{4 - x^2}$ but both of these solutions are only valid on the interval $[-2, 2]$ which ends at 2, and neither of these solutions can be extended past 2).

Example 30. Consider the initial value problem $(x^2 - 1)y' + \sin(xy) = x^2$, $y(a) = b$. For which values of a and b can we guarantee existence and uniqueness of a (local) solution?

Solution. Let us write $y' = f(x, y)$ with $f(x, y) = \frac{x^2 - \sin(xy)}{x^2 - 1}$. Then $\frac{\partial}{\partial y} f(x, y) = \frac{-\cos(xy)}{x^2 - 1} \cdot (y + xy')$.

Both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $x^2 \neq 1$ which is equivalent to $x \neq \pm 1$.

Hence, if $a \neq \pm 1$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Example 31. Consider the initial value problem $y' = y^{1/3}$, $y(a) = b$. For which values of a and b can we guarantee existence and uniqueness of a (local) solution?

Solution. Let us write $y' = f(x, y)$ with $f(x, y) = y^{1/3}$. Then $\frac{\partial}{\partial y} f(x, y) = \frac{1}{3} y^{-2/3}$.

While $f(x, y) = y^{1/3}$ is always continuous, $\frac{\partial}{\partial y} f(x, y) = \frac{1}{3} y^{-2/3}$ is only continuous for all (x, y) with $y \neq 0$.

Hence, if $b \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Challenge. Solve the DE as well as plot the slope field. Then analyze what we can say about solutions in the case $b = 0$. (This is a case where we get existence but not uniqueness. It illustrates that an extra condition like continuity of $\frac{\partial}{\partial y} f(x, y)$ is needed.)

Armin Straub
straub@southalabama.edu

ODEs vs PDEs

Important. Note that we are working with functions $y(x)$ of a single variable. This allows us to write simply y' for $\frac{\mathrm{d}}{\mathrm{d}x}y(x)$ without risk of confusion.

Of course, we may use different variables such as $x(t)$ and $x' = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, as long as this is clear from the context.

Differential equations that involve only derivatives with respect to a single variable are known as **ordinary differential equations** (ODEs).

On the other hand, differential equations that involve derivatives with respect to several variables are referred to as **partial differential equations** (PDEs).

Example 32. The DE

$$\left(\frac{\partial}{\partial x}\right)^2 u(x, y) + \left(\frac{\partial}{\partial y}\right)^2 u(x, y) = 0,$$

often abbreviated as $u_{xx} + u_{yy} = 0$, is a partial differential equation in two variables.

This particular PDE is known as **Laplace's equation** and describes, for ialsonstance, steady-state heat distributions.

https://en.wikipedia.org/wiki/Laplace%27s_equation

This and other fundamental PDEs will be discussed in Differential Equations II.

Linear first-order DEs

A **linear differential equation** is one where the function y and its derivatives only show up linearly (i.e. there are no terms such as y^2, $1/y$, $\sin(y)$ or $y \cdot y'$).

As such, the most general linear first-order DE is of the form

$$A(x)y' + B(x)y + C(x) = 0.$$

Such a DE can be rewritten in the following "**standard form**" by dividing by $A(x)$ and rearranging:

(linear first-order DE in standard form)

$$y' + P(x)y = Q(x)$$

We will use this standard form when solving linear first-order DEs.

Example 33. (extra "warmup") Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2$.

Solution. (separation of variables) $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, $-\frac{1}{y} = x^2 + C$.

Hence the general solution is $y = \frac{1}{D - x^2}$. [There also is the singular solution $y = 0$.]

Solution. (in other words) Note that $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ can be written as $\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{1}{y}\right] = \frac{\mathrm{d}}{\mathrm{d}x}[x^2]$.

From there it follows that $-\frac{1}{y} = x^2 + C$, as above.

We now use the idea of writing both sides as a derivative (which we then integrate!) to also solve DEs that are not separable. We will be able to handle all first-order linear DEs this way.

The multiplication by $\frac{1}{y^2}$ will be replaced by multiplication with a so-called **integrating factor**.

Example 34. Solve $y' = x - y$.

Comment. Note that we cannot use separation of variables this time.

Solution. Rewrite the DE as $y' + y = x$.

Next, multiply both sides with e^x (we will see in a little bit how to find this "integrating factor") to get

$$\underbrace{e^x y' + e^x y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^x y]} = x e^x.$$

The "magic" part is that we are able to realize the new left-hand side as a derivative!

We can then integrate both sides to get

$$e^x y = \int x e^x \mathrm{d}x = x e^x - e^x + C.$$

From here it follows that $y = x - 1 + Ce^{-x}$.

Comment. For the final integral, we used that $\int x e^x \mathrm{d}x = x e^x - \int e^x \mathrm{d}x = x e^x - e^x + C$ which follows, for instance, via integration by parts (with $f(x) = x$ and $g'(x) = e^x$ in the formula reviewed below).

Review. The product rule $(fg)' = f' g + f g'$ implies $fg = \int f' g + \int f g'$.

The latter is equivalent to **integration by parts**:

$$\int f(x) g'(x) \mathrm{d}x = f(x) g(x) - \int f'(x) g(x) \mathrm{d}x$$

Comment. Sometimes, one writes $g'(x) \mathrm{d}x = \mathrm{d}g(x)$.

Armin Straub
straub@southalabama.edu

In general, we can solve any **linear first-order DE** $y' + P(x)y = Q(x)$ in this way.

- We want to multiply with an **integrating factor** $f(x)$ such that the left-hand side of the DE becomes

$$f(x)y' + f(x)P(x)y = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)y].$$

  Since $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)y] = f(x)y' + f'(x)y$, we need $f'(x) = f(x)P(x)$ for that.

- Check that $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$ has this property.

  Comment. This follows directly from computing the derivative of this $f(x)$ via the chain rule.

  Homework. On the other hand, note that finding f meant solving the DE $f' = P(x)\, f$. This is a separable DE. Solve it by separation of variables to arrive at the above formula for $f(x)$ yourself.

  Just to make sure. There is no difference between $\exp(x)$ and e^x. Here, we prefer the former notation for typographical reasons.

With that integrating factor, we have the following recipe for solving any linear first-order equation:

(solving linear first-order DEs)

(a) Write the DE in the **standard form** $y' + P(x)y = Q(x)$.

(b) Compute the **integrating factor** as $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$.

[We can choose any constant of integration.]

(c) Multiply the DE from part (a) by $f(x)$ to get

$$\underbrace{f(x)y' + f(x)P(x)y}_{=\frac{\mathrm{d}}{\mathrm{d}x}[f(x)y]} = f(x)Q(x).$$

(d) Integrate both sides to get

$$f(x)y = \int f(x)Q(x)\mathrm{d}x + C.$$

Then solve for y by dividing by $f(x)$.

Comment. For better understanding, we prefer to go through the above steps. On the other hand, we can combine these steps into the following formula for the general solution of $y' + P(x)y = Q(x)$:

$$y = \frac{1}{f(x)}\left(\int f(x)Q(x)\mathrm{d}x + C\right) \quad \text{where } f(x) = e^{\int P(x)\mathrm{d}x}$$

Existence and uniqueness. Note that the solution we construct exists on any interval on which P and Q are continuous (not just on some possibly very small interval). This is better than what the existence and uniqueness theorem (Theorem 23) can guarantee. This is one of the many ways in which linear DEs have particularly nice properties compared to DEs in general.

Armin Straub
straub@southalabama.edu

Example 35. Solve $xy' = 2y + 1$, $y(-2) = 0$.

Solution. This is a linear first-order DE.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ (standard form) with $P(x) = -\frac{2}{x}$ and $Q(x) = \frac{1}{x}$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{-2\ln|x|} = e^{-2\ln(-x)} = (-x)^{-2} = \frac{1}{x^2}$.

Here, we used that, at least locally, $x < 0$ (because the initial condition is $x = -2 < 0$) so that $|x| = -x$.

(c) Multiply the DE (in standard form) by $f(x) = \frac{1}{x^2}$ to get

$$\underbrace{\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x^3}y}_{= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x^2}y\right]} = \frac{1}{x^3}.$$

(d) Integrate both sides to get

$$\frac{1}{x^2}y = \int \frac{1}{x^3}\mathrm{d}x = -\frac{1}{2x^2} + C.$$

Hence, the general solution is $y(x) = -\frac{1}{2} + Cx^2$.

Solving $y(-2) = -\frac{1}{2} + 4C = 0$ for C yields $C = \frac{1}{8}$. Thus, the (unique) solution to the IVP is $y(x) = \frac{1}{8}x^2 - \frac{1}{2}$.

Example 36. (extra) Solve $y' = 2y + 3x - 1$, $y(0) = 2$.

Solution. This is a linear first-order DE.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ (standard form) with $P(x) = -2$ and $Q(x) = 3x - 1$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{-2x}$.

(c) Multiply the DE (in standard form) by $f(x) = e^{-2x}$ to get

$$\underbrace{e^{-2x}\frac{\mathrm{d}y}{\mathrm{d}x} - 2e^{-2x}y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^{-2x}y]} = (3x-1)e^{-2x}.$$

(d) Integrate both sides to get

$$\begin{aligned} e^{-2x}y &= \int (3x-1)e^{-2x}\mathrm{d}x \\ &= 3\int xe^{-2x}\mathrm{d}x - \int e^{-2x}\mathrm{d}x \\ &= 3\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right) - \left(-\frac{1}{2}e^{-2x}\right) + C \\ &= -\frac{3}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C. \end{aligned}$$

Here, we used that $\int xe^{-2x}\mathrm{d}x = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\mathrm{d}x = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$ (for instance, via integration by parts with $f(x) = x$ and $g'(x) = e^{-2x}$).

Hence, the general solution is $y(x) = -\frac{3}{2}x - \frac{1}{4} + Ce^{2x}$.

Solving $y(0) = -\frac{1}{4} + C = 2$ for C yields $C = \frac{9}{4}$.

In conclusion, the (unique) solution to the IVP is $y(x) = -\frac{3}{2}x - \frac{1}{4} + \frac{9}{4}e^{2x}$.

Armin Straub
straub@southalabama.edu

Review. We can solve linear first-order DEs using **integrating factors**.

First, put the DE in standard form $y' + P(x)y = Q(x)$. Then $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$ is the integrating factor.

The key is that we get on the left-hand side $f(x)y' + f(x)P(x)y = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)\,y]$. We can therefore integrate both sides with respect to x (the right-hand side is $f(x)Q(x)$ which is just a function depending on x—not y!).

Example 37. Solve $x^2 y' = 1 - xy + 2x$, $y(1) = 3$.

Solution. This is a linear first-order DE. We can therefore solve it according to the recipe above.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ (standard form) with $P(x) = \frac{1}{x}$ and $Q(x) = \frac{1}{x^2} + \frac{2}{x}$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{\ln x} = x$.

Here, we could write $\ln x$ instead of $\ln|x|$ because the initial condition tells us that $x > 0$, at least locally.
Comment. We can also choose a different constant of integration but that would only complicate things.

(c) Multiply the DE (in standard form) by $f(x) = x$ to get

$$\underbrace{x\frac{\mathrm{d}y}{\mathrm{d}x} + y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[xy]} = \frac{1}{x} + 2.$$

(d) Integrate both sides to get (again, we use that $x > 0$ to avoid having to use $|x|$)

$$xy = \int \left(\frac{1}{x} + 2\right)\mathrm{d}x = \ln x + 2x + C.$$

Using $y(1) = 3$ to find C, we get $1 \cdot 3 = \ln(1) + 2 \cdot 1 + C$ which results in $C = 3 - 2 = 1$.

Hence, the (unique) solution to the IVP is $y = \frac{\ln(x) + 2x + 1}{x}$.

Substitutions in DEs

Example 38. (review) Using substitution, compute $\int \frac{x}{1 + x^2}\mathrm{d}x$.

Solution. We substitute $u = 1 + x^2$. In that case, $\mathrm{d}u = 2x\,\mathrm{d}x$.
$\int \frac{x}{1 + x^2}\mathrm{d}x = \frac{1}{2}\int \frac{1}{u}\mathrm{d}u = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(1 + x^2) + C$

Comment. Why were we allowed to drop the absolute value in the logarithm?

Review. On the other hand, recall that $\int \frac{1}{1 + x^2}\mathrm{d}x = \arctan(x) + C$.

Example 39. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = (x + y)^2$.

First things first. Is this DE separable? Is it linear? (No to both but make sure that this is clear to you.)
This means that our previous techniques are not sufficient to solve this DE.

Solution. Looking at the right-hand side, we have a feeling that the substitution $u = x + y$ might simplify things.
Then $y = u - x$ and, therefore, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - 1$.

Using these, the DE translates into $\frac{\mathrm{d}u}{\mathrm{d}x} - 1 = u^2$. This is a separable DE: $\frac{1}{1 + u^2}\mathrm{d}u = \mathrm{d}x$

After integration, we find $\arctan(u) = x + C$ and, thus, $u = \tan(x + C)$.

The solution of the original DE is $y = u - x = \tan(x + C) - x$.

Armin Straub
straub@southalabama.edu

Useful substitutions

The previous example illustrates that different substitutions can help to solve a given DE.

Choosing the right substitution is difficult in general. The following is a compilation of important cases that are easy to spot and for which the listed substitutions are guaranteed to succeed:

- $y' = F\left(\frac{y}{x}\right)$

  Set $u = \frac{y}{x}$. Then $y = ux$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$. We get $x\frac{\mathrm{d}u}{\mathrm{d}x} + u = F(u)$. This DE is always separable.

  Caution. The DE $y' = F\left(\frac{y}{x}\right)$ is sometimes called a "homogeneous equation". However, we will soon discuss homogeneous linear differential equations, where the label homogeneous means something different (though in both cases, there is a common underlying reason).

- $y' = F(ax + by)$

  Set $u = ax + by$. Then $y = \frac{1}{b}(u - ax)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{b}\left(\frac{\mathrm{d}u}{\mathrm{d}x} - a\right)$.

  The new DE is $\frac{1}{b}\left(\frac{\mathrm{d}u}{\mathrm{d}x} - a\right) = F(u)$ or, simplified, $\frac{\mathrm{d}u}{\mathrm{d}x} = a + bF(u)$. This DE is always separable.

- $y' = F(x)y + G(x)y^n$ (This is called a **Bernoulli equation**.)

  Set $u = y^{1-n}$. The resulting DE is always linear.

  Details. If $u = y^{1-n}$ then $y = u^{1/(1-n)}$ and, thus, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-n}u^{n/(1-n)}\frac{\mathrm{d}u}{\mathrm{d}x}$. $\left[\frac{1}{1-n} - 1 = \frac{n}{1-n}\right]$

  The new DE is $\frac{1}{1-n}u^{n/(1-n)}\frac{\mathrm{d}u}{\mathrm{d}x} = F(x)u^{1/(1-n)} + G(x)u^{n/(1-n)}$.

  Dividing both sides by $u^{n/(1-n)}$, the DE simplifies to $\frac{1}{1-n}\frac{\mathrm{d}u}{\mathrm{d}x} = F(x)u + G(x)$ which is a linear DE.

  Comment. The original DE has the trivial solution $y = 0$. Do you see where we lost that solution?

Example 40. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = (2x - 3y)^2 + \frac{2}{3}$, $y(1) = \frac{1}{3}$.

Solution. This is of the form $y' = F(2x - 3y)$ with $F(t) = t^2 + \frac{2}{3}$.

Therefore, as suggested by our list of useful substitutions, we substitute $u = 2x - 3y$.

Then $y = \frac{1}{3}(2x - u)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}\left(2 - \frac{\mathrm{d}u}{\mathrm{d}x}\right)$.

The new DE is $\frac{1}{3}\left(2 - \frac{\mathrm{d}u}{\mathrm{d}x}\right) = u^2 + \frac{2}{3}$ or, simplified, $\frac{\mathrm{d}u}{\mathrm{d}x} = -3u^2$.

This DE is separable: $u^{-2}\mathrm{d}u = -3\mathrm{d}x$. After integration, $-\frac{1}{u} = -3x + C$.

We conclude that $u = \frac{1}{3x - C}$ and, hence, $y(x) = \frac{1}{3}(2x - u) = \frac{2}{3}x - \frac{1}{3}\frac{1}{3x - C}$.

Solving $y(1) = \frac{2}{3} - \frac{1}{3}\frac{1}{3 - C} = \frac{1}{3}$ for C leads to $C = 2$.

Hence, the unique solution of the IVP is $y(x) = \frac{2}{3}x - \frac{1}{3(3x - 2)}$.

Example 41. (homework) Consider the DE $x\frac{\mathrm{d}y}{\mathrm{d}x} = y + y^2 f(x)$.

(a) Substitute $u = \frac{y}{x}$. Is the resulting DE separable or linear?

(b) Substitute $v = \frac{1}{y}$. Is the resulting DE separable or linear?

(c) Solve each of the new DEs.

Solution.

(a) Set $u = \frac{y}{x}$. Then $y = ux$ and, thus, $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$.

Using these, the DE translates into $x\left(x\frac{\mathrm{d}u}{\mathrm{d}x} + u\right) = ux + (ux)^2 f(x)$.

This DE simplifies to $\frac{\mathrm{d}u}{\mathrm{d}x} = u^2 f(x)$. This is a separable DE.

(b) Set $v = \frac{1}{y}$. Then $y = \frac{1}{v}$ and, thus, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x}$.

Using these, the DE translates into $x\left(-\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x}\right) = \frac{1}{v} + \frac{1}{v^2} f(x)$.

This DE simplifies to $x\frac{\mathrm{d}v}{\mathrm{d}x} = -v - f(x)$. This is a linear DE.

(c) Let us write $F(x)$ for an antiderivative of $f(x)$.

- The DE $\frac{\mathrm{d}u}{\mathrm{d}x} = u^2 f(x)$ from the first part is separable: $u^2 \mathrm{d}u = f(x)\mathrm{d}x$.
  After integration, we find $-\frac{1}{u} = F(x) + C$.
  Since $u = \frac{y}{x}$, this becomes $-\frac{x}{y} = F(x) + C$.
  The general solution of the initial DE therefore is $y = -\frac{x}{F(x) + C}$.

- The DE $x\frac{\mathrm{d}v}{\mathrm{d}x} = -v - f(x)$ from the second part is linear. We apply our recipe:

  (a) Rewrite the DE as $\frac{\mathrm{d}v}{\mathrm{d}x} + P(x)v = Q(x)$ with $P(x) = 1/x$ and $Q(x) = -f(x)/x$.

  (b) The integrating factor is $\exp\left(\int P(x)\mathrm{d}x\right) = e^{\ln x} = x$.

  Comment. We should make a mental note that we assumed that $x > 0$. In the next step, however, we see that the integrating factor works for all x.

  (c) Multiply the (rewritten) DE by the integrating factor x to get $\underbrace{x\frac{\mathrm{d}v}{\mathrm{d}x} + v}_{= \frac{\mathrm{d}}{\mathrm{d}x}[xv]} = -f(x)$.

  (d) Integrate both sides to get $xv = -F(x) + C$.

  Since $v = \frac{1}{y}$, we find $\frac{x}{y} = -F(x) + C$.
  The general solution of the initial DE therefore is $y = -\frac{x}{F(x) - C}$.

Comment. Note that our two approaches led to the same general solution (from the existence and uniqueness theorem, we can see that this must be the case). One of the formulas features $+C$ while the other features $-C$. However, that makes no difference because C is a free parameter (we could have given them different names if we preferred).

Example 42. Solve $(x-y)\frac{dy}{dx} = x+y$.

Solution. Divide the DE by x to get $\left(1-\frac{y}{x}\right)\frac{dy}{dx} = 1+\frac{y}{x}$. This is a DE of the form $y' = F\left(\frac{y}{x}\right)$.

We therefore substitute $u = \frac{y}{x}$. Then $y = ux$ and $\frac{dy}{dx} = x\frac{du}{dx} + u$.

The resulting DE is $(x-ux)\left(x\frac{du}{dx}+u\right) = x+ux$, which simplifies to $x(1-u)\frac{du}{dx} = 1+u^2$.

This DE is separable: $\frac{1-u}{1+u^2}du = \frac{1}{x}dx$

Integrating both sides, we find $\arctan(u) - \frac{1}{2}\ln(1+u^2) = \ln|x| + C$.

Setting $u = y/x$, we get the (general) implicit solution $\arctan(y/x) - \frac{1}{2}\ln(1+(y/x)^2) = \ln|x| + C$.

Comment. We used $\int \frac{1}{1+u^2}du = \arctan(u) + C$ and $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2) + C$ when integrating.

See Example 38 where we reviewed these integrals.

Example 43. Solve the IVP $\frac{dy}{dx} = 2y - 3xy^5$, $y(0) = 1$.

Solution. This is an example of a Bernoulli equation (with $n=5$). We therefore substitute $u = y^{1-n} = y^{-4}$.

Accordingly, $y = u^{-1/4}$ and, thus, $\frac{dy}{dx} = -\frac{1}{4}u^{-5/4}\frac{du}{dx}$.

The new DE is $-\frac{1}{4}u^{-5/4}\frac{du}{dx} = 2u^{-1/4} - 3xu^{-5/4}$, which simplifies to $\frac{du}{dx} = -8u + 12x$.

This is a linear first-order DE, which we solve according to our recipe:

(a) Rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 8$ and $Q(x) = 12x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^{8x}$.

(c) Multiply the (rewritten) DE by $f(x) = e^{8x}$ to get

$$\underbrace{e^{8x}\frac{du}{dx} + 8e^{8x}u}_{=\frac{d}{dx}[e^{8x}u]} = 12xe^{8x}.$$

(d) Integrate both sides to get:

$$e^{8x}u = 12\int xe^{8x}dx = 12\left(\frac{1}{8}xe^{8x} - \frac{1}{8^2}e^{8x}\right) + C = \frac{3}{2}xe^{8x} - \frac{3}{16}e^{8x} + C$$

Here we used that $\int xe^{ax}dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}$. (Integration by parts!)

The general solution of the DE for u therefore is $u = \frac{3}{2}x - \frac{3}{16} + Ce^{-8x}$.

Correspondingly, the general solution of the initial DE is $y = u^{-1/4} = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + Ce^{-8x}}$.

Using $y(0) = 1$, we find $1 = 1/\sqrt[4]{C - \frac{3}{16}}$ from which we obtain $C = 1 + \frac{3}{16} = \frac{19}{16}$.

The unique solution to the IVP therefore is $y = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + \frac{19}{16}e^{-8x}}$.

Armin Straub
straub@southalabama.edu

Solving simple 2nd order DEs

We have the following two useful substitutions for certain simple DEs of order 2:

- $F(y'', y', x) = 0$ (2nd order with "y missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx}$. We get the first-order DE $F\left(\frac{du}{dx}, u, x\right) = 0$.

- $F(y'', y', y) = 0$ (2nd order with "x missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$. We get the first-order DE $F\left(u\frac{du}{dy}, u, y\right) = 0$.

Example 44. Solve $y'' = x - y'$.

Solution. We substitute $u = y'$, which results in the first-order DE $u' = x - u$.
This DE is linear and, using our recipe (see below for the details), we can solve it to find $u = x - 1 + Ce^{-x}$.
Since $y' = u$, we conclude that the general solution is $y = \int (x - 1 + Ce^{-x})dx = \frac{1}{2}x^2 - x - Ce^{-x} + D$.

Important comment. This is a DE of order 2. Hence, as expected, the general solution has two free parameter.

Solving the linear DE. To solve $u' = x - u$ (also see Example 34, where we had solved this DE before), we

(a) rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 1$ and $Q(x) = x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^x$.

(c) Multiply the (rewritten) DE by $f(x) = e^x$ to get $\underbrace{e^x\frac{du}{dx} + e^x u}_{= \frac{d}{dx}[e^x u]} = xe^x$.

(d) Integrate both sides to get (using integration by parts): $e^x u = \int xe^x dx = xe^x - e^x + C$

Hence, the general solution of the DE for u is $u = x - 1 + Ce^{-x}$, which is what we used above.

Example 45. (homework) Solve the IVP $y'' = x - y'$, $y(0) = 1$, $y'(0) = 2$.

Solution. As in the previous example, we find that the general solution to the DE is $y(x) = \frac{1}{2}x^2 - x - Ce^{-x} + D$.
Using $y'(x) = x - 1 + Ce^{-x}$ and $y'(0) = 2$, we find that $2 = -1 + C$. Hence, $C = 3$.
Then, using $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + D$ and $y(0) = 1$, we find $1 = -3 + D$. Hence, $D = 4$.
In conclusion, the unique solution to the IVP is $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + 4$.

Example 46. (extra) Find the general solution to $y'' = 2yy'$.

Solution. We substitute $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$.
Therefore, our DE turns into $u\frac{du}{dy} = 2yu$.
Dividing by u, we get $\frac{du}{dy} = 2y$. [Note that we lose the solution $u = 0$, which gives the singular solution $y = C$.]
Hence, $u = y^2 + C$. It remains to solve $y' = y^2 + C$. This is a separable DE.
$\frac{1}{C + y^2}dy = dx$. Let us restrict to $C = D^2 \geqslant 0$ here. (This means we will only find "half" of the solutions.)
$\int \frac{1}{D^2 + y^2}dy = \frac{1}{D^2}\int \frac{1}{1 + (y/D)^2}dy = \frac{1}{D}\arctan(y/D) = x + A$.
Solving for y, we find $y = D\tan(Dx + AD) = D\tan(Dx + B)$. $[B = AD]$

Armin Straub
straub@southalabama.edu