Review. To model a falling object, we let $y(t)$ be its height at time t. If we only take earth's gravitation into account, then the fall is modeled by

$$y''(t) = -g$$

where the gravitational acceleration is $g \approx 9.81 \mathrm{m/s}^2$.

For many applications, one needs to take air resistance into account.

This is actually less well understood than one might think, and the physics quickly becomes rather complicated. Typically, air resistance is somewhere in between the following two cases:

- Under certain assumptions, physics suggests that air resistance is proportional to the square of the velocity.

  Comment. A simplistic way to think about this is to imagine the falling object to bump into (air) particles; if the object falls twice as fast, then the momentum of the particles it bumps into is twice as large and it bumps into twice as many of them.

- In other cases such as "relatively slowly" falling objects, one might empirically observe that air resistance is proportional to the velocity itself.

  Comment. One might imagine that, at slow speed, the falling object doesn't exactly bump into particles but instead just gently pushes them aside; so that at twice the speed it only needs to gently push twice as often.

Example 59. When modeling the (slow) fall of a parachute, physics suggests that the air resistance is roughly proportional to velocity. If $y(t)$ is the parachute's height at time t, then the corresponding DE is $y'' = -g - \rho y'$ where $\rho > 0$ is a constant.

Comment. Note that $-\rho y' > 0$ because $y' < 0$. Thus, as intended, air resistance is acting in the opposite direction as gravity and slowing down the fall.

Determine the general solution of the DE.

Solution. Substituting $v = y'$, the DE becomes $v' + \rho v = -g$.

This is a linear DE. To solve it, we determine that the integrating factor is $\exp(\int \rho \mathrm{d}t) = e^{\rho t}$.

Multiplying the DE with that factor and integrating, we obtain $e^{\rho t} v = \int -g e^{\rho t} \mathrm{d}t = -\frac{g}{\rho} e^{\rho t} + C$.

Hence, $v(t) = -\frac{g}{\rho} + C e^{-\rho t}$.

Correspondingly, the general solution of the DE is $y(t) = \int v(t) \mathrm{d}t = -\frac{g}{\rho} t - \frac{C}{\rho} e^{-\rho t} + D$.

Comment. Note that $\lim_{t \to \infty} v(t) = -\frac{g}{\rho}$. In other words, the **terminal velocity** is $v_\infty = -\frac{g}{\rho}$.

This is an interesting mathematical consequence of the DE. (And important for the idea behind a parachute!)

Note that, if we know that there is a terminal speed, then we can actually determine its value v_∞ from the DE without solving it by setting $v' = 0$ (because, once the terminal speed is reached, the velocity does not change anymore) in $v' + \rho v = -g$. This gives us $\rho v_\infty = -g$ and, hence, $v_\infty = -g / \rho$ as above.

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Numerically "solving" DEs: Euler's method

Recall that the general form of a first-order initial value problem is

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Further recall that, under mild assumptions on $f(x, y)$, such an IVP has a unique solution $y(x)$.
We have learned some techniques for (exactly) solving DEs. On the other hand, many DEs that arise in practice cannot be solved by these techniques (or more fancy ones).

Instead, it is common in practice to approximate the solution $y(x)$ to our IVP. Euler's method is the simplest example of how this can done. The key idea is to locally approximate $y(x)$ by tangent lines:

Example 60. Suppose y solves the IVP $y' = f(x, y)$ with $y(x_0) = y_0$. Using the tangent line at (x_0, y_0), find an approximation for $y(x_1)$ where $x_1 = x_0 + h$.

The idea is that we choose the **step size** h to be sufficiently small so that the approximation is good enough.

Solution. The tangent line at (x_0, y_0) has slope $y'(x_0) = f(x_0, y_0)$ which is a number we can compute without solving the DE for $y(x)$. Hence, the equation for the tangent line is $T(x) = y_0 + f(x_0, y_0)(x - x_0)$.

We now use this tangent line as an approximation for the solution of the DE to find

$$y(x_1) \approx T(x_1) = y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + f(x_0, y_0)h.$$

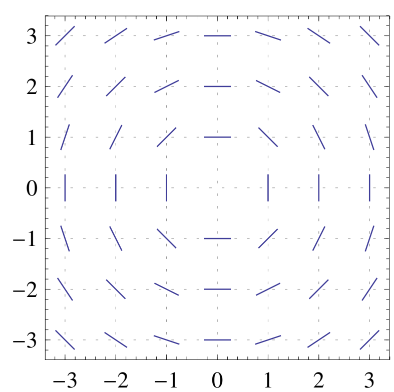
At this point, we have gone from our initial point (x_0, y_0) to a next (approximate) point (x_1, y_1).
We now repeat what we did to get a third point (x_2, y_2) with $x_2 = x_1 + h$. Continuing in this way, we obtain Euler's method:

(Euler's method) To approximate the solution $y(x)$ of the IVP $y' = f(x, y)$, $y(x_0) = y_0$, we start with the point (x_0, y_0) and a step size h. We then compute:

$$\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + h f(x_n, y_n) \end{aligned}$$

Example 61. Consider, again, the DE $y' = -x/y$.

We earlier produced the slope field on the right. We also used separation of variables to find that the solutions are circles $y(x) = \pm\sqrt{r^2 - x^2}$.



We know that the unique solution to the IVP with $y(0) = 2$ is $y(x) = \sqrt{4 - x^2}$. On the other hand, approximate the solution using Euler's method with $h = 1$ and 2 steps.

Solution. First, use just the slope field to sketch the 2 approximate points.
On the other hand, applying Euler's method with $f(x, y) = -x/y$:

$$\begin{aligned} &x_0 = 0 \quad && y_0 = 2 \\ &x_1 = 1 && y_1 = y_0 + h f(x_0, y_0) = 2 + 1 \cdot \left(-\frac{0}{2}\right) = 2 \\ &x_2 = 2 && y_2 = y_1 + h f(x_1, y_1) = 2 + 1 \cdot \left(-\frac{1}{2}\right) = \frac{3}{2} \end{aligned}$$

Comment. These are not good approximations! (To get better approximations, the step size must be chosen much smaller.) For comparison, the true values are $y(1) = \sqrt{3} \approx 1.73$ and $y(2) = 0$. Also note that we would get "bogus" values if we take another step to approximate $y(3)$ (whereas the true solution only exists until $x = 2$).

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Example 62. Consider the IVP $\frac{\mathrm{d}y}{\mathrm{d}x} = (2x-3y)^2 + \frac{2}{3}$, $y(1) = \frac{1}{3}$.

(a) Approximate the solution $y(x)$ for $x \in [1,2]$ using Euler's method with 2 steps.

(b) Approximate the solution $y(x)$ for $x \in [1,2]$ using Euler's method with 3 steps.

(c) Solve this IVP exactly. Compare the values at $x = 2$.

Solution.

(a) The step size is $h = \frac{2-1}{2} = \frac{1}{2}$. We apply Euler's method with $f(x,y) = (2x-3y)^2 + \frac{2}{3}$:

$$\begin{array}{ll} x_0 = 1 & y_0 = \frac{1}{3} \\ x_1 = \frac{3}{2} & y_1 = y_0 + h f(x_0, y_0) = \frac{1}{3} + \frac{1}{2} \cdot \left[\left(2 \cdot 1 - 3 \cdot \frac{1}{3}\right)^2 + \frac{2}{3}\right] = \frac{7}{6} \\ x_2 = 2 & y_2 = y_1 + h f(x_1, y_1) = \frac{7}{6} + \frac{1}{2} \cdot \frac{11}{12} = \frac{13}{8} \end{array}$$

In particular, the approximation for $y(2)$ is $y_2 = \frac{13}{8} = 1.625$.

(b) The step size is $h = \frac{2-1}{3} = \frac{1}{3}$. We again apply Euler's method with $f(x,y) = (2x-3y)^2 + \frac{2}{3}$:

$$\begin{array}{ll} x_0 = 1 & y_0 = \frac{1}{3} \\ x_1 = \frac{4}{3} & y_1 = y_0 + h f(x_0, y_0) = \frac{1}{3} + \frac{1}{3} \cdot \left[\left(2 \cdot 1 - 3 \cdot \frac{1}{3}\right)^2 + \frac{2}{3}\right] = \frac{8}{9} \\ x_2 = \frac{5}{3} & y_2 = y_1 + h f(x_1, y_1) = \frac{8}{9} + \frac{1}{3} \cdot \frac{2}{3} = \frac{10}{9} \\ x_3 = 2 & y_3 = y_2 + h f(x_2, y_2) = \frac{10}{9} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{3} \end{array}$$

In particular, the approximation for $y(2)$ is $y_3 = \frac{4}{3} \approx 1.333$.

(c) We solved this IVP in Example 40 using the substitution $u = 2x - 3y$ followed by separation of variables. We found that the unique solution of the IVP is $y(x) = \frac{2}{3}x - \frac{1}{3(3x-2)}$.

In particular, the exact value at $x = 2$ is $y(2) = \frac{5}{4} = 1.25$.

We observe that our approximations for $y(2) = 1.25$ improved from 1.625 to 1.333 as we increased the number of steps (equivalently, we decreased the step size h from $\frac{1}{2}$ to $\frac{1}{3}$).

For comparison. With 10 steps (so that $h = \frac{1}{10}$), the approximation improves to $y(2) \approx 1.259$.

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