Review. Mixing problems

Example 57. (follow-up) A tank contains 20gal of water mixed with an unknown amount of salt. It is filled with brine (containing 5lb/gal salt) at a rate of 3gal/min. At the same time, well-mixed solution flows out at a rate of 2gal/min. Approximately, how much salt is in the tank after t minutes if t is large?

Solution. Clearly, the tank will contain $V(t) = 20 + (3-2)t = 20 + t$ gal after t minutes.

For large t, the initial 20gal will be sufficiently diluted by the incoming brine so that the concentration of salt in the tank will approach 5lb/gal. This means that, after t min, the tank contains about $5V(t) = 5(20+t)$ lb of salt (if t is large).

Solution. (via DE which is extra work that is unnecessary here) Alternatively, we could proceed as in the previous example: determine a DE for $x(t)$ (the lb of salt in the tank after t min) and then solve it. The only difference is that we don't have the initial condition $x(0) = 0$ (since the amount of salt is unknown). We still find that the general solution is $x(t) = 5(20+t) + \frac{C}{(20+t)^2}$. We can then observe that we always have $x(t) \approx 5(20+t)$ for large t (no matter the value of C).

Application: Acceleration–velocity models

To model a falling object, we let $y(t)$ be its height at time t.

Then physics has names for $y'(t)$ and $y''(t)$: these are the **velocity** and the **acceleration**.

Physics tells us that objects fall due to gravity (and that it makes already-falling objects fall faster; in other words, gravity accelerates falling objects). Physicists have measured that, on earth, the gravitational acceleration is $g \approx 9.81 \mathrm{m/s}^2$.

If we only take earth's gravitation into account, then the fall is therefore modeled by

$$y''(t) = -g.$$

Example 58. A ball is dropped from a 100m tall building. How long until it reaches the ground? What is the speed when it hits the ground?

Solution. Let $y(t)$ be the height (in meters) at which the ball is at time t (in seconds).

As above, physics tells us that an object falling due to gravity (and ignoring everything else) satisfies the DE $y'' = -g$ where $g \approx 9.81$. We further know the initial values $y(0) = 100$, $y'(0) = 0$.

Substituting $v = y'$ in the DE, we get $v' = -g$. This DE is solved by $v(t) = -gt + C$.

Hence, $y(t) = \int v(t) \mathrm{d}t = -\frac{1}{2}gt^2 + Ct + D$.

The initial conditions $y(0) = 100$, $y'(0) = 0$ tell us that $D = 100$ and $C = 0$.

Thus $y(t) = -\frac{1}{2}gt^2 + 100$.

The ball reaches the ground when $y(t) = -\frac{1}{2}gt^2 + 100 = 0$, that is after $t = \sqrt{200/g} \approx 4.52$ seconds.

The speed then is $|y'(4.52)| \approx 44.3 \, \mathrm{m}/s$.

Armin Straub
straub@southalabama.edu