**Example 55.** A biotech company is growing certain microorganisms in the lab. From experience they know that the growth (number of organisms per day) of the microorganisms is well modeled by an exponential model with proportionality constant $k=5$ (per day). What is the optimal rate (in number of organisms per day) at which the company can continually harvest the microorganisms?

**Solution. (long version via solving the DE)** Without harvesting, the growth is modeled by $\frac{dP}{dt}=5P$ (the exponential model). Here, $P$ is the number of organisms and $t$ measures time in days. (Always think about your units in applications!)

If harvesting occurs at the rate of $h$ number of organisms per day, the population model needs to be adjusted to

$$\frac{dP}{dt}=5P-h.$$

Since $h$ is a constant, we can solve this DE using separation of variables. Alternatively, the DE is linear and we can therefore solve it using an integrating factor. For practice, we do both:

- **(separation of variables)** Integrating $\frac{1}{5P-h}dP=dt$, we find $\frac{1}{5}\ln|5P-h|=t+C$, which we simplify to $|5P-h|=e^{5t+5C}$. It follows that $5P-h=\pm e^{5t}e^{5C}=Be^{5t}$ where we wrote $B=\pm e^{5C}$ (note that the sign is fixed and cannot change).
  Thus, the general solution of the DE is $P(t)=\frac{h}{5}+Ae^{5t}$ (where we wrote $A=\frac{B}{5}$).
- **(integrating factor)** Since this is a linear DE, we can solve it as follows:
  - We write the DE in the form $\frac{dP}{dt}-5P=-h$.
  - The integrating factor is $f(t)=\exp(\int -5dt)=e^{-5t}$.
  - Multiply the (rewritten) DE by $f(t)$ to get $\underbrace{e^{-5t}\frac{dP}{dt}-5e^{-5t}P}_{=\frac{d}{dt}[e^{-5t}P]}=-he^{-5t}$.
  - Integrate both sides to get $e^{-5t}P=-h\int e^{-5t}dt=\frac{h}{5}e^{-5t}+C$.

  Hence the general solution to the DE is $P(t)=\frac{h}{5}+Ce^{5t}$.

In either case, we found that $P(t)=\frac{h}{5}+Ce^{5t}$. In order to be able to continually harvest, we need to make sure that $C\geqslant 0$. In terms of the initial population, we get $P(0)=\frac{h}{5}+C$ so that $C=P(0)-\frac{h}{5}$.

Thus the condition $C\geqslant 0$ becomes $P(0)-\frac{h}{5}\geqslant 0$ or, equivalently, $h\leqslant 5P(0)$. Thus, the optimal rate of harvesting is $h=5P(0)$.

**Solution. (short version)** As before, we observe that, if harvesting occurs at the rate of $h$ number of organisms per day, then our population model is

$$\frac{dP}{dt}=5P-h.$$

The crucial observation is that the optimal harvesting rate should occur if $\frac{dP}{dt}=0$ (if $\frac{dP}{dt}>0$, then the population grows indicating that we could have harvested at a higher rate; if $\frac{dP}{dt}<0$ then the population shrinks and, all else being equal, we should no longer be able to harvest at the optimal rate).

We thus get the condition $5P-h=0$. Since the population is constant (because of $\frac{dP}{dt}=0$) this is equivalent to $5P(0)-h=0$. Again, we conclude that the optimal rate of harvesting is $h=5P(0)$.

# Application: Mixing problems

**Example 56.** A tank contains 20gal of pure water. It is filled with brine (containing 5lb/gal salt) at a rate of 3gal/min. At the same time, well-mixed solution flows out at a rate of 2gal/min. How much salt is in the tank after $t$ minutes?

**Solution.**

**(Part I. determining a DE)** Let $x(t)$ denote the amount of salt (in lb) in the tank after time $t$ (in min).

At time $t$, the concentration of salt (in lb/gal) in the tank is $\frac{x(t)}{V(t)}$ where $V(t) = 20 + (3-2)t = 20 + t$ is the volume (in gal) in the tank.

In the time interval $[t, t + \Delta t]$: $\Delta x \approx 3 \cdot 5 \cdot \Delta t - 2 \cdot \frac{x(t)}{V(t)} \cdot \Delta t$.

Hence, $x(t)$ solves the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = 15 - 2 \cdot \frac{x}{20+t}$ with $x(0) = 0$.

**Comment.** Can you explain why the equation for $\Delta x$ is only approximate but why the final DE is exact?
[Hint: $x(t)/V(t)$ is the concentration at time $t$ but we are using it for $\Delta x$ at other times as well.]

**(Part II. solving the DE)** Since this is a linear DE, we can solve it as follows:

- Write the DE in the standard form as $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2}{20+t} x = 15$.
- The integrating factor is $f(t) = \exp\left(\int \frac{2}{20+t} \mathrm{d}t\right) = \exp(2 \ln|20+t|) = (20+t)^2$.
- Multiply the DE (in standard form) by $f(t) = (20+t)^2$ to get $\underbrace{(20+t)^2 \frac{\mathrm{d}x}{\mathrm{d}t} + 2(20+t)x}_{= \frac{\mathrm{d}}{\mathrm{d}t}[(20+t)^2 x]} = 15(20+t)^2$.
- Integrate both sides to get $(20+t)^2 x = 15 \int (20+t)^2 \mathrm{d}t = 5(20+t)^3 + C$.

Hence the general solution to the DE is $x(t) = 5(20+t) + \frac{C}{(20+t)^2}$. Using $x(0) = 0$, we find $C = -5 \cdot 20^3$.

We conclude that, after $t$ minutes, the tank contains $x(t) = 5(20+t) - \frac{5 \cdot 20^3}{(20+t)^2}$ pounds of salt.

**Comment.** As a consequence, $x(t) \approx 5(20+t) = 5V(t)$ for large $t$. Can you explain why that makes perfect sense and why we could have predicted this from the very beginning (without deriving a DE and solving it)?

Armin Straub
straub@southalabama.edu