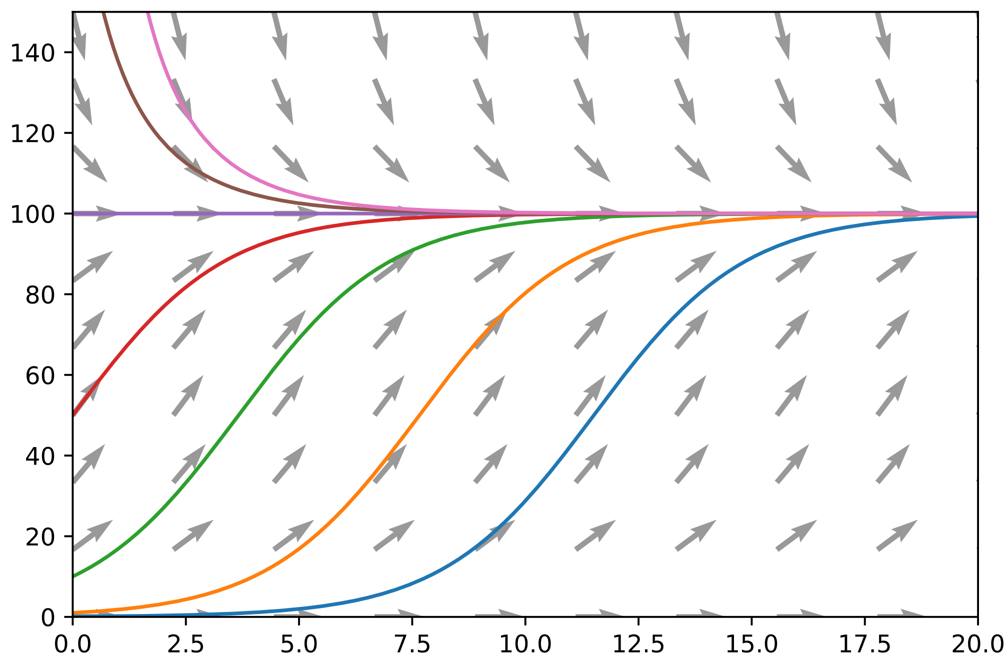
Review. The **logistic equation** is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$.

Here, k is the growth rate and M is the carrying capacity.

The general solution of the logistic equation is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

Example 52. Visualize the logistic equation for $k = 0.6$ and $M = 100$ using a slope field as well as by plotting some solution functions.

Solution.



In this slope field, we plotted the solutions $P(t)$ with $P(0) = 0.1, 1, 10, 50, 100, 200, 1000$.

Main challenge of modeling: a model has to be detailed enough to resemble the real world, yet simple enough to allow for mathematical analysis.

Extending the exponential model. Observe that the exponential model of population growth can be written as

$$\frac{P'}{P} = \text{constant}.$$

Thinking purely mathematically (generally not a good idea for modeling!), to extend the model, it might be sensible to replace constant (which we called k above) by the next simplest kind of function, namely a linear function in P. The resulting DE is the **logistic equation**.

Comment. Can you put into words why we replace constant by a function of P rather than a function of t? When would it be appropriate to add a dependence on t?

[A dependence on t would make sense if the "environment" changes over time. Without such a change, we expect that a population (say, of bacteria in our lab) behaves this week just as it would next week. The "law" behind its growth should not depend on t. The resulting differential equations are called **autonomous**.]

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Example 53. In a city with a fixed population N, the time rate of change of the number P of people who have heard a certain rumor is proportional to the product of P and $N - P$. Suppose initially 10% have heard the rumor and after a week this number has grown to 20%. What percentage will this number reach after one more week?

Solution. We are told that $\frac{\mathrm{d}P}{\mathrm{d}t} = \gamma P(N - P)$ as well as $P(0) = 0.1N$ and $P(1) = 0.2N$. We need $P(2)$.

Note that this is a logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$ with $k = \gamma N$ and carrying capacity N.

It therefore has the general solution $P(t) = \dfrac{N}{1 + Ce^{-kt}}$.

Using $P(0) = \frac{N}{1 + C} = 0.1N$, we find that $C = 9$.

Using $P(1) = \frac{N}{1 + 9e^{-k}} = 0.2N$, we further find that $e^{-k} = \frac{4}{9}$.

We could solve for k but note that it is more pleasing to use $e^{-kt} = (e^{-k})^t = \left(\frac{4}{9}\right)^t$ in our formula for $P(t)$.

We conclude that $P(t) = \dfrac{N}{1 + 9\left(\frac{4}{9}\right)^t}$.

In particular, $P(2) = \frac{N}{1 + 9 \cdot \frac{16}{81}} = \frac{9}{25}N$ which is 36%.

Example 54. A scientist is claiming that a certain population $P(t)$ follows the logistic model of population growth. How many data points do you need to begin to verify that claim?

Solution. The general solution $P(t) = \dfrac{M}{1 + Ce^{-kt}}$ to the logistic equation has 3 parameters.

Hence, we need 3 data points just to solve for their values.

Once we have 4 or more data points, we are able to test whether $P(t)$ conforms to the logistic model.

Important comment. Complicated models tend to have more degrees of freedom, which makes it easier to fit them to real world data (even if the model is not actually particularly appropriate). We therefore need to be cognizant about how much evidence is needed to decide that a given model is appropriate for the data.

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Further population models

Let $P(t)$ be the size of the population that we wish to model at time t.

Denote with $\beta(t)$ and $\delta(t)$ the birth and death rate at time t, measured in number of births or deaths per unit of population per unit of time.

In the time interval $[t, t + \Delta t]$, we have that

$$\Delta P \approx \beta(t)P(t)\Delta t - \delta(t)P(t)\Delta t.$$

Comment. The reason that this is not an exact equation is that the rates $\beta(t)$ and $\delta(t)$ are allowed to change with t. In the above, we used these rates at time t for all times in $[t, t + \Delta t]$. This is a good approximation if Δt is small.

Divide both sides by Δt and let $\Delta t \to 0$ to obtain the general differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P.$$

Given certain scenarios, we now make corresponding reasonable choices for $\beta(t)$ and $\delta(t)$.

- **(basic)** If the rates $\beta(t)$ and $\delta(t)$ are constant over time, the DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta - \delta)P$.
  This is the exponential model of population growth.

- **(limited supply)** If supply is limited, the birth rate will decrease as P increases. The simplest such relationship would be a linear dependence, which would take the form $\beta(t) = \beta_0 - \beta_1 P$.
  On the other hand, we still assume that $\delta(t)$ is constant. (However, depending on circumstances, it could also be reasonable to assume that $\delta(t)$ increases as P increases.)
  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta_0 - \beta_1 P - \delta)P$.
  This is the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(1 - P/M)$ with $k = \beta_0 - \delta$ and $\frac{k}{M} = \beta_1$.

- **(rare isolated species)** If the population consists of rare and isolated specimen which rely on chance encounters to reproduce, then it is reasonable to assume that the birth rate $\beta(t)$ is proportional to $P(t)$ (larger $P(t)$ means more possibilities for chance encounters). Once more, we assume that $\delta(t)$ constant.
  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (kP - \delta)P$.
  This is, again, the logistic equation.

- **(rare isolated species with very long life)** As before, for a rare isolated population, it is reasonable to assume that $\beta(t)$ is proportional to $P(t)$. If, in addition, our specimen have very long life, then we would assume that $\delta(t) = 0$.
  The corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP^2$. Solutions are $P(t) = \frac{1}{C - kt}$ where $P(0) = 1/C$. (Do it!)
  Comment. Note that $P(t) \to \infty$ as $t \to C/k$. This explosion (which implies population growth beyond exponential growth) emphasizes that we can only use the DE while our initial assumptions are satisfied. Here, the DE is no longer appropriate when our species is no longer rare because $P(t)$ is too large.

- **(spread of contagious incurable virus)** Let $P(t)$ count the number of infected population units among a (constant) total of N. Since the virus is incurable, we have $\delta(t) = 0$. On the other hand, it is reasonable to assume that $\beta(t)$ is proportional to $N - P$ (the number of people that can still be infected).
  The resulting DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(N - P)$. Once again, this is the logistic equation.

- **(harvesting)** Suppose that h population units are harvested each unit of time.
  Then the DE becomes $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P - h$.
  For instance. $\frac{\mathrm{d}P}{\mathrm{d}t} = kP - h$ has the solution $P(t) = Ce^{kt} + h/k$. In that case, we get exponential growth if $C > 0$. Note that $P(0) = C + h/k$. In terms of the initial population $P(0)$, we therefore get exponential growth if $P(0) > h/k$. (Also see next example!)

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