

Example 42. Solve $(x - y)\frac{dy}{dx} = x + y$.

Solution. Divide the DE by x to get $(1 - \frac{y}{x})\frac{dy}{dx} = 1 + \frac{y}{x}$. This is a DE of the form $y' = F(\frac{y}{x})$.

We therefore substitute $u = \frac{y}{x}$. Then $y = ux$ and $\frac{dy}{dx} = x\frac{du}{dx} + u$.

The resulting DE is $(x - ux)(x\frac{du}{dx} + u) = x + ux$, which simplifies to $x(1 - u)\frac{du}{dx} = 1 + u^2$.

This DE is separable: $\frac{1 - u}{1 + u^2} du = \frac{1}{x} dx$

Integrating both sides, we find $\arctan(u) - \frac{1}{2}\ln(1 + u^2) = \ln|x| + C$.

Setting $u = y/x$, we get the (general) implicit solution $\arctan(y/x) - \frac{1}{2}\ln(1 + (y/x)^2) = \ln|x| + C$.

Comment. We used $\int \frac{1}{1 + u^2} du = \arctan(u) + C$ and $\int \frac{x}{1 + x^2} dx = \frac{1}{2}\ln(1 + x^2) + C$ when integrating.

See Example 38 where we reviewed these integrals.

Example 43. Solve the IVP $\frac{dy}{dx} = 2y - 3xy^5$, $y(0) = 1$.

Solution. This is an example of a Bernoulli equation (with $n = 5$). We therefore substitute $u = y^{1-n} = y^{-4}$.

Accordingly, $y = u^{-1/4}$ and, thus, $\frac{dy}{dx} = -\frac{1}{4}u^{-5/4}\frac{du}{dx}$.

The new DE is $-\frac{1}{4}u^{-5/4}\frac{du}{dx} = 2u^{-1/4} - 3xu^{-5/4}$, which simplifies to $\frac{du}{dx} = -8u + 12x$.

This is a linear first-order DE, which we solve according to our recipe:

(a) Rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 8$ and $Q(x) = 12x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^{8x}$.

(c) Multiply the (rewritten) DE by $f(x) = e^{8x}$ to get

$$\begin{aligned} e^{8x}\frac{du}{dx} + 8e^{8x}u &= 12xe^{8x} \\ \underbrace{\hspace{1.5cm}} &= \frac{d}{dx}[e^{8x}u] \end{aligned}$$

(d) Integrate both sides to get:

$$e^{8x}u = 12 \int xe^{8x}dx = 12\left(\frac{1}{8}xe^{8x} - \frac{1}{8^2}e^{8x}\right) + C = \frac{3}{2}xe^{8x} - \frac{3}{16}e^{8x} + C$$

Here we used that $\int xe^{ax}dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}$. (Integration by parts!)

The general solution of the DE for u therefore is $u = \frac{3}{2}x - \frac{3}{16} + Ce^{-8x}$.

Correspondingly, the general solution of the initial DE is $y = u^{-1/4} = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + Ce^{-8x}}$.

Using $y(0) = 1$, we find $1 = 1/\sqrt[4]{C - \frac{3}{16}}$ from which we obtain $C = 1 + \frac{3}{16} = \frac{19}{16}$.

The unique solution to the IVP therefore is $y = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + \frac{19}{16}e^{-8x}}$.

Solving simple 2nd order DEs

We have the following two useful substitutions for certain simple DEs of order 2:

- $F(y'', y', x) = 0$ (2nd order with “ y missing”)
Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx}$. We get the first-order DE $F\left(\frac{du}{dx}, u, x\right) = 0$.
- $F(y'', y', y) = 0$ (2nd order with “ x missing”)
Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$. We get the first-order DE $F\left(u \frac{du}{dy}, u, y\right) = 0$.

Example 44. Solve $y'' = x - y'$.

Solution. We substitute $u = y'$, which results in the first-order DE $u' = x - u$.

This DE is linear and, using our recipe (see below for the details), we can solve it to find $u = x - 1 + Ce^{-x}$.

Since $y' = u$, we conclude that the general solution is $y = \int (x - 1 + Ce^{-x}) dx = \frac{1}{2}x^2 - x - Ce^{-x} + D$.

Important comment. This is a DE of order 2. Hence, as expected, the general solution has two free parameter.

Solving the linear DE. To solve $u' = x - u$ (also see Example 34, where we had solved this DE before), we

- (a) rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 1$ and $Q(x) = x$.
- (b) The integrating factor is $f(x) = \exp\left(\int P(x) dx\right) = e^x$.
- (c) Multiply the (rewritten) DE by $f(x) = e^x$ to get $\underbrace{e^x \frac{du}{dx} + e^x u}_{= \frac{d}{dx}[e^x u]} = xe^x$.
- (d) Integrate both sides to get (using integration by parts): $e^x u = \int xe^x dx = xe^x - e^x + C$

Hence, the general solution of the DE for u is $u = x - 1 + Ce^{-x}$, which is what we used above.

Example 45. (homework) Solve the IVP $y'' = x - y'$, $y(0) = 1$, $y'(0) = 2$.

Solution. As in the previous example, we find that the general solution to the DE is $y(x) = \frac{1}{2}x^2 - x - Ce^{-x} + D$.

Using $y'(x) = x - 1 + Ce^{-x}$ and $y'(0) = 2$, we find that $2 = -1 + C$. Hence, $C = 3$.

Then, using $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + D$ and $y(0) = 1$, we find $1 = -3 + D$. Hence, $D = 4$.

In conclusion, the unique solution to the IVP is $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + 4$.

Example 46. (extra) Find the general solution to $y'' = 2yy'$.

Solution. We substitute $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$.

Therefore, our DE turns into $u \frac{du}{dy} = 2yu$.

Dividing by u , we get $\frac{du}{dy} = 2y$. [Note that we lose the solution $u = 0$, which gives the singular solution $y = C$.]

Hence, $u = y^2 + C$. It remains to solve $y' = y^2 + C$. This is a separable DE.

$\frac{1}{C + y^2} dy = dx$. Let us restrict to $C = D^2 \geq 0$ here. (This means we will only find “half” of the solutions.)

$\int \frac{1}{D^2 + y^2} dy = \frac{1}{D^2} \int \frac{1}{1 + (y/D)^2} dy = \frac{1}{D} \arctan(y/D) = x + A$.

Solving for y , we find $y = D \tan(Dx + AD) = D \tan(Dx + B)$.

[$B = AD$]