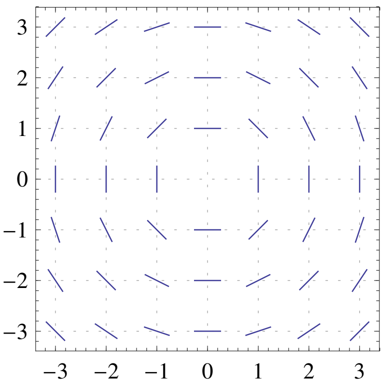
Review. Existence and uniqueness theorem (Theorem 23) for an IVP $y' = f(x, y)$, $y(a) = b$:
If $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous around (a, b) then, locally, the IVP has a unique solution.

Example 29. Consider, again, the IVP $y' = -x / y$, $y(a) = b$. Discuss existence and uniqueness of solutions (without solving).



Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = -x / y$.

We compute that $\frac{\partial}{\partial y} f(x, y) = x / y^2$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $y \neq 0$.

Hence, if $b \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Comment. In Example 14, we found that the DE $y' = -x / y$ is solved by $y(x) = \pm\sqrt{D - x^2}$.

Assume $b > 0$ (things work similarly for $b < 0$). Then $y(x) = \sqrt{D - x^2}$ solves the IVP (we need to choose D so that $y(a) = b$) if we choose $D = a^2 + b^2$. This confirms that there exists a solution. On the other hand, uniqueness means that there can be no other solution to the IVP than this one.

What happens in the case $b = 0$?

Solution. In this case, the existence and uniqueness theorem does not guarantee anything. If $a \neq 0$, then $y(x) = \sqrt{a^2 - x^2}$ and $y(x) = -\sqrt{a^2 - x^2}$ both solve the IVP (so we certainly don't have uniqueness), however only in a weak sense: namely, both of these solutions are not valid locally around $x = a$ but only in an interval of which a is an endpoint (for instance, the IVP $y' = -x / y$, $y(2) = 0$ is solved by $y(x) = \pm\sqrt{4 - x^2}$ but both of these solutions are only valid on the interval $[-2, 2]$ which ends at 2, and neither of these solutions can be extended past 2).

Example 30. Consider the initial value problem $(x^2 - 1)y' + \sin(xy) = x^2$, $y(a) = b$. For which values of a and b can we guarantee existence and uniqueness of a (local) solution?

Solution. Let us write $y' = f(x, y)$ with $f(x, y) = \frac{x^2 - \sin(xy)}{x^2 - 1}$. Then $\frac{\partial}{\partial y} f(x, y) = \frac{-\cos(xy)}{x^2 - 1} \cdot (y + xy')$.

Both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $x^2 \neq 1$ which is equivalent to $x \neq \pm 1$.

Hence, if $a \neq \pm 1$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Example 31. Consider the initial value problem $y' = y^{1/3}$, $y(a) = b$. For which values of a and b can we guarantee existence and uniqueness of a (local) solution?

Solution. Let us write $y' = f(x, y)$ with $f(x, y) = y^{1/3}$. Then $\frac{\partial}{\partial y} f(x, y) = \frac{1}{3} y^{-2/3}$.

While $f(x, y) = y^{1/3}$ is always continuous, $\frac{\partial}{\partial y} f(x, y) = \frac{1}{3} y^{-2/3}$ is only continuous for all (x, y) with $y \neq 0$.

Hence, if $b \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Challenge. Solve the DE as well as plot the slope field. Then analyze what we can say about solutions in the case $b = 0$. (This is a case where we get existence but not uniqueness. It illustrates that an extra condition like continuity of $\frac{\partial}{\partial y} f(x, y)$ is needed.)

Armin Straub
straub@southalabama.edu

ODEs vs PDEs

Important. Note that we are working with functions $y(x)$ of a single variable. This allows us to write simply y' for $\frac{\mathrm{d}}{\mathrm{d}x}y(x)$ without risk of confusion.

Of course, we may use different variables such as $x(t)$ and $x' = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, as long as this is clear from the context.

Differential equations that involve only derivatives with respect to a single variable are known as **ordinary differential equations** (ODEs).

On the other hand, differential equations that involve derivatives with respect to several variables are referred to as **partial differential equations** (PDEs).

Example 32. The DE

$$\left(\frac{\partial}{\partial x}\right)^2 u(x,y) + \left(\frac{\partial}{\partial y}\right)^2 u(x,y) = 0,$$

often abbreviated as $u_{xx} + u_{yy} = 0$, is a partial differential equation in two variables.

This particular PDE is known as **Laplace's equation** and describes, for ialsonstance, steady-state heat distributions.

https://en.wikipedia.org/wiki/Laplace%27s_equation

This and other fundamental PDEs will be discussed in Differential Equations II.

Linear first-order DEs

A **linear differential equation** is one where the function y and its derivatives only show up linearly (i.e. there are no terms such as y^2, $1/y$, $\sin(y)$ or $y \cdot y'$).

As such, the most general linear first-order DE is of the form

$$A(x)y' + B(x)y + C(x) = 0.$$

Such a DE can be rewritten in the following "**standard form**" by dividing by $A(x)$ and rearranging:

(linear first-order DE in standard form)

$$y' + P(x)y = Q(x)$$

We will use this standard form when solving linear first-order DEs.

Example 33. (extra "warmup") Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2$.

Solution. (separation of variables) $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, $-\frac{1}{y} = x^2 + C$.

Hence the general solution is $y = \frac{1}{D - x^2}$. [There also is the singular solution $y = 0$.]

Solution. (in other words) Note that $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ can be written as $\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{1}{y}\right] = \frac{\mathrm{d}}{\mathrm{d}x}[x^2]$.

From there it follows that $-\frac{1}{y} = x^2 + C$, as above.

We now use the idea of writing both sides as a derivative (which we then integrate!) to also solve DEs that are not separable. We will be able to handle all first-order linear DEs this way.

The multiplication by $\frac{1}{y^2}$ will be replaced by multiplication with a so-called **integrating factor**.

Example 34. Solve $y' = x - y$.

Comment. Note that we cannot use separation of variables this time.

Solution. Rewrite the DE as $y' + y = x$.

Next, multiply both sides with e^x (we will see in a little bit how to find this "integrating factor") to get

$$\underbrace{e^x y' + e^x y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^x y]} = x e^x.$$

The "magic" part is that we are able to realize the new left-hand side as a derivative!

We can then integrate both sides to get

$$e^x y = \int x e^x \mathrm{d}x = x e^x - e^x + C.$$

From here it follows that $y = x - 1 + Ce^{-x}$.

Comment. For the final integral, we used that $\int x e^x \mathrm{d}x = x e^x - \int e^x \mathrm{d}x = x e^x - e^x + C$ which follows, for instance, via integration by parts (with $f(x) = x$ and $g'(x) = e^x$ in the formula reviewed below).

Review. The product rule $(fg)' = f'g + fg'$ implies $fg = \int f'g + \int fg'$.

The latter is equivalent to **integration by parts**:

$$\int f(x) g'(x) \mathrm{d}x = f(x) g(x) - \int f'(x) g(x) \mathrm{d}x$$

Comment. Sometimes, one writes $g'(x) \mathrm{d}x = \mathrm{d}g(x)$.

Armin Straub
straub@southalabama.edu