

In the following example, we first proceed like we did when producing a slope field to compute slopes (and, therefore, tangent lines) of solutions. Indeed, besides the slope y' , we can compute further derivatives like y'' or y''' by differentiating the DE.

Do you recall how y'' tells us about the curvature of a function $y(x)$?

Example 17. Consider the DE $x^2y' = 1 + xy^3$. Suppose that $y(x)$ is a solution passing through the point $(2, 1)$.

Important. This is the same as saying that $y(x)$ solves the IVP $x^2y' = 1 + xy^3$ with $y(2) = 1$.

- Determine $y'(2)$.
- Determine the tangent line of $y(x)$ at $(2, 1)$.
- Determine $y''(2)$.

Comment. Note that this DE is not separable.

Solution.

- At the point $(2, 1)$ we have $x = 2$ and $y = 1$. Plugging these values into the differential equation, we get $4y' = 1 + 2 \cdot 1^3 = 3$ which we can solve for y' to find $y' = \frac{3}{4}$.

Since y' is short for $y'(x) = y'(2)$, we have found $y'(2) = \frac{3}{4}$.

- The tangent line is the line through $(2, 1)$ with slope $\frac{3}{4}$ (computed in the previous part).

From this information, we can immediately write down its equation in the form $y = \frac{3}{4}(x - 2) + 1$.

- To get our hands on $y''(2)$, we can differentiate (with respect to x) both sides of $x^2y' = 1 + xy^3$.

Applying the product rule, we have $\frac{d}{dx}x^2y'(x) = 2xy'(x) + x^2y''(x) = 2xy' + x^2y''$ as well as $\frac{d}{dx}(1 + xy(x)^3) = y(x)^3 + x \cdot 3y(x)^2 \cdot y'(x) = y^3 + 3xy^2y'$.

Thus $2xy' + x^2y'' = y^3 + 3xy^2y'$. To find $y''(2)$, we plug in $x = 2$, $y = 1$, $y' = \frac{3}{4}$.

This results in $2 \cdot 2 \cdot \frac{3}{4} + 4y'' = 1 + 3 \cdot 2 \cdot 1 \cdot \frac{3}{4}$ or $3 + 4y'' = \frac{11}{2}$. It follows that $y'' = \frac{1}{4} \cdot \frac{5}{2} = \frac{5}{8}$.

Comment. Alternatively, we can rewrite the DE as $y' = \frac{1}{x^2} + \frac{1}{x}y^3$ and then differentiate. Do it!

Comment. Do you recall from Calculus what it means visually to have $y'' = \frac{5}{8}$?

[Since $y'' > 0$ it means that our function is concave up at $(2, 1)$. As such, its graph will lie above the tangent line.]

Comment. Note that we could continue and likewise find $y'''(2)$ or higher derivatives at $x = 2$. This is the starting point for the power series method typically discussed in Differential Equations II.

Example 18. (warm-up) Consider the DE $xy' = 2y$.

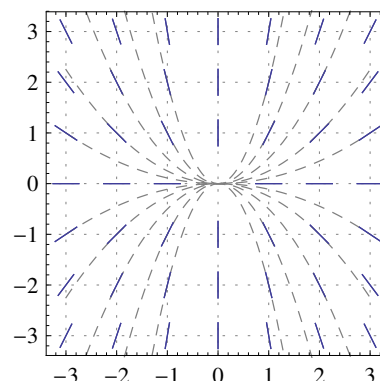
Sketch its slope field.

Challenge. Try to guess solutions $y(x)$ from the slope field.

Solution. For instance, to find the slope at the point $(3, 1)$, we plug $x = 3$, $y = 1$ into the DE to get $3y' = 2$. Hence, the slope is $y' = 2/3$.

The resulting slope field is sketched on the right.

Solution of the challenge. Trace out the solution through $(1, 1)$ (and then some other points). Their shape looks like a parabola, so that we might guess that $y(x) = Cx^2$ solves the DE. Check that this is indeed the case by plugging into the DE!



Solving DEs: Separation of variables, cont'd

In general, **separation of variables** solves $y' = g(x)h(y)$ by writing the DE as $\frac{1}{h(y)} dy = g(x) dx$.

Note that $\frac{1}{h(y)} \frac{dy}{dx} = g(x)$ is indeed equivalent to $\int \frac{1}{h(y)} dy = \int g(x) dx + C$. Why?! (Apply $\frac{d}{dx}$ to the integrals...)

Example 19. (cont'd) Solve the IVP $xy' = 2y$, $y(1) = 2$.

Solution. Rewrite the DE as $\frac{1}{y} \frac{dy}{dx} = \frac{2}{x}$.

Then multiply both sides with dx and integrate both of them to get $\int \frac{1}{y} dy = \int \frac{2}{x} dx$.

Hence, $\ln|y| = 2\ln|x| + C$.

The initial condition $y(1) = 2$ tells us that, at least locally, $x > 0$ and $y > 0$. Thus $\ln(y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = 2$, we find $C = \ln(2)$.

Solving $\ln(y) = 2\ln(x) + \ln(2)$ for y , we find $y = e^{2\ln(x) + \ln(2)} = 2x^2$.

Comment. When solving a DE or IVP, we can generally only expect to find a **local solution**, meaning that our solution might only be valid in a small interval around the initial condition (here, we can only expect $y(x)$ to be a solution for all x in an interval around 1; especially since we assumed $x > 0$ in our solution). However, we can check (do it!) that the solution $y = 2x^2$ is actually a **global solution** (meaning that it is a solution for all x , not just locally around 1).

Advanced comment. However, there are other global solutions such as $y(x) = \begin{cases} 2x^2, & \text{for } x \geq 0, \\ 7x^2, & \text{for } x < 0. \end{cases}$

Can you see that this piecewise defined function $y(x)$ is differentiable everywhere? (The slope at $x = 0$ is 0.)

The 7 can, of course, be replaced with any other number.

Let's solve the same differential equation with a different choice of initial condition:

Example 20. Solve the IVP $xy' = 2y$, $y(1) = -1$.

Solution. Again, we rewrite the DE as $\frac{1}{y} \frac{dy}{dx} = \frac{2}{x}$, multiply both sides with dx , and integrate to get $\int \frac{1}{y} dy = \int \frac{2}{x} dx$.

Hence, $\ln|y| = 2\ln|x| + C$. The initial condition $y(1) = -1$ tells us that, at least locally, $x > 0$ and $y < 0$ (note that this means $|y| = -y$). Thus $\ln(-y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = -1$, we find $C = 0$.

Solving $\ln(-y) = 2\ln(x)$ for y , we find $y = -e^{2\ln(x)} = -x^2$. We easily verify that this is indeed a global solution.

Example 21. $y' = x + y$ is a DE for which the variables cannot be separated.

No worries, very soon we will have several tools to solve this DE as well.