Midterm #1

MATH 481/581 — Cryptography
Wednesday, Feb 26

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 40 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (6 points) Using the Chinese remainder theorem, determine all solutions to $x^2 \equiv 4 \pmod{55}$.

Solution. By the CRT:

$$\begin{aligned} & x^2 \equiv 4 \pmod{55} \\ \iff & x^2 \equiv 4 \pmod{5} \text{ and } x^2 \equiv 4 \pmod{11} \\ \iff & x \equiv \pm 2 \pmod{5} \text{ and } x \equiv \pm 2 \pmod{11} \end{aligned}$$

Hence, there are four solutions $\pm 2, \pm a$ modulo 55. To find one of the nontrivial ones, we solve the congruences $x \equiv 2 \pmod{5}$, $x \equiv -2 \pmod{11}$:

$$x \equiv 2 \cdot 11 \cdot \underbrace{11^{-1}_{\bmod 5}}_{1} - 2 \cdot 5 \cdot \underbrace{5^{-1}_{\bmod 11}}_{-2} \equiv 22 + 20 \equiv -13 \pmod{55}$$

Hence, we conclude that $x^2 \equiv 4 \pmod{55}$ has the four solutions $\pm 2, \pm 13 \pmod{55}$.

Problem 2. (4 points)

(a) Suppose N is composite. x is a Fermat liar modulo N if and only if ☐.

(b) $8 \pmod{21}$ ☐ is a Fermat liar ☐ is not a Fermat liar because ☐.

Solution.

(a) x is a Fermat liar modulo N if and only if $x^{N-1} \equiv 1 \pmod{N}$.

(b) 8 is a Fermat liar modulo 21 if and only if $8^{20} \equiv 1 \pmod{21}$.

$8^2 \equiv 1 \pmod{21}$, so that $8^{20} \equiv 1 \pmod{21}$. Hence, 8 a Fermat liar modulo 21.

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Problem 3. (7 points) Eve intercepts the ciphertext $c = (000\ 111\ 000)_2$. She knows it was encrypted with a stream cipher using the linear congruential generator $x_{n+1} \equiv 5x_n + 1 \pmod{8}$ as PRG.

Eve also knows that the plaintext begins with $m = (011\ 1\ldots)_2$. Break the cipher and determine the plaintext.

Solution. Since $c = m \oplus \mathrm{PRG}$, we learn that the initial piece of the keystream is $\mathrm{PRG} = c \oplus m = (000\ 111\ 000)_2 \oplus (011\ 1\ldots)_2 = (011\ 0\ldots)_2$.

Since each x_n has 3 bits, we learn that $x_1 = (011)_2 = 3$. Using $x_{n+1} \equiv 5x_n + 1 \pmod{8}$, we find $x_2 = 0$, $x_3 = 1$, ... In other words, $\mathrm{PRG} = 3, 0, 1, \ldots = (011\ 000\ 001\ \ldots)_2$.

Hence, Eve can decrypt the ciphertext and obtain $m = c \oplus \mathrm{PRG} = (000\ 111\ 000)_2 \oplus (011\ 000\ 001)_2 = (011\ 111\ 001)_2$.

Problem 4. (5 points) Evaluate $23^{16013} \pmod{17}$. Show your work!

Solution. First, $23^{16013} \equiv 6^{16013} \pmod{17}$. Since $16013 \equiv 13 \pmod{\phi(17)}$, we have $6^{16013} \equiv 6^{13} \pmod{17}$.

Using binary exponentiation, we find $6^2 \equiv 2 \pmod{17}$, $6^4 \equiv 2^2 = 4 \pmod{17}$, $6^8 \equiv 4^2 \equiv -1 \pmod{17}$.

In conclusion, $23^{16013} \equiv 6^{13} = 6^8 \cdot 6^4 \cdot 6 \equiv -1 \cdot 4 \cdot 6 \equiv 10 \pmod{17}$.

Problem 5. (3 points) Briefly outline the Fermat primality test.

Solution. Fermat primality test:

Input: number n and parameter k indicating the number of tests to run
Output: “not prime” or “possibly prime”
Algorithm:

Repeat k times:
    Pick a random number a from $\{2, 3, \ldots, n-2\}$.
    If $a^{n-1} \not\equiv 1 \pmod{n}$, then stop and output “not prime”.
Output “possibly prime”.

Problem 6. (15 points) Fill in the blanks.

(a) The residue x is invertible modulo n if and only if [ ].

(b) $3^{-1} \pmod{29} \equiv$ [ ].

(c) Modulo 29, there are [ ] invertible residues, of which [ ] are quadratic.

(d) Modulo 55, there are [ ] invertible residues, of which [ ] are quadratic.

(e) 24 in base 2 is [ ].

(f) How many solutions does the congruence $x^2 \equiv 1 \pmod{105}$ have? [ ]

How many solutions does the congruence $x^2 \equiv 9 \pmod{105}$ have? [ ]

(g) Despite its flaws, in which scenario is it fine to use the Fermat primality test?

[ ]

(h) The first 5 bits generated by the Blum-Blum-Shub PRG with $M = 133$ using the seed 5 are [ ].

You may use that $16^2 \equiv 123$, $25^2 \equiv 93$, $36^2 \equiv 99$, $92^2 \equiv 85$, $93^2 \equiv 4$, $99^2 \equiv 92 \pmod{133}$.

(i) Using a one-time pad and key $k = (0011)_2$, the message $m = (1010)_2$ is encrypted to [ ].

(j) While perfectly confidential, the one-time pad does not protect against [ ].

(k) The LFSR $x_{n+31} \equiv x_{n+28} + x_n \pmod{2}$ must repeat after [ ] terms.

(l) Recall that, in a stream cipher, we must never reuse the key stream.

Nevertheless, we can reuse the key if we use a [ ].

(m) In order for a PRG to be suitable for use in a stream cipher, the PRG must be [ ].

(n) As part of the Miller–Rabin test, it is computed that $26^{147} \equiv 495$, $26^{294} \equiv 1 (\mathrm{mod}\, 589)$.

What do we conclude? | |

(o) Up to x, there are roughly | | many primes.

Solution.

(a) The residue x is invertible modulo n if and only if $\gcd(x, n) = 1$.

(b) $3^{-1} \pmod{29} \equiv 10$.

(c) Modulo the prime 29, there are $\phi(29) = 28$ invertible residues, of which $\frac{1}{2}\phi(29) = 14$ are quadratic.

(d) Modulo 55, there are $\phi(55) = \phi(5)\phi(11) = 40$ invertible residues, of which $\frac{1}{4}\phi(55) = 10$ are quadratic.

(e) 24 in base 2 is $(11000)_2$.

(f) By the CRT, since $105 = 3 \cdot 5 \cdot 7$, the first congruence has $2 \cdot 2 \cdot 2 = 8$ solutions.

The second congruence only has $1 \cdot 2 \cdot 2 = 4$ solutions. (Note that $x^2 \equiv 9 \pmod{3}$ only has one solution; namely, $x \equiv 0$.)

(g) Despite its flaws, it is fine to use the Fermat primality test for large random numbers.

(h) The first five bits generated by the Blum-Blum-Shub PRG with $M = 133$ using the seed 5 are $1, 1, 0, 0, 1$ (obtained from $25, 93, 4, 16, 123$).

(i) Using a one-time pad and key $k = (0011)_2$, the message $m = (1010)_2$ is encrypted to $(1001)_2$.

(j) While perfectly confidential, the one-time pad does not protect against tampering.

(k) The LFSR $x_{n+31} \equiv x_{n+28} + x_n \pmod{2}$ must repeat after $2^{31} - 1$ terms.

(l) We can reuse the key if we use a nonce.

(m) In order for a PRG to be suitable for use in a stream cipher, the PRG must be unpredictable.

(n) Since $495 \not\equiv -1 \pmod{589}$, we conclude that 589 is not a prime.

(o) Up to x, there are roughly $x / \ln(x)$ many primes.

(extra scratch paper)