- Recall that, in contrast to DES, the operations of AES have very simple (though somewhat advanced) mathematical descriptions.

  No mysteriously constructed S-boxes and P-boxes as in DES.

## ByteSub (continued)

Each of the 16 bytes gets substituted as follows.

**Note.** The mathematical description below can be implemented in a **lookup table**: you can find this table in Table 5.1 of our book or, for instance, on wikipedia: https://en.wikipedia.org/wiki/Rijndael_S-box

- Interpret the input byte $(b_7b_6...b_0)_2$ as the element $b_7x^7+...+b_1x+b_0$ of $\mathrm{GF}(2^8)$.
- Compute $s^{-1}=c_0+c_1x+...+c_7x^7$ (with $0^{-1}$ interpreted as $0$).

  **Important comment.** This inversion is what makes AES highly nonlinear.

  If the ByteSub substitution was linear, then all of AES would be linear (because all other layers are linear; assuming we adjust the key schedule accordingly).

- Then the output bits $(d_7d_6...d_1d_0)_2$ are

$$\begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_7 \end{bmatrix} = \begin{bmatrix} 1&0&0&0&1&1&1&1 \\ 1&1&0&0&0&1&1&1 \\ 1&1&1&0&0&0&1&1 \\ 1&1&1&1&0&0&0&1 \\ 1&1&1&1&1&0&0&0 \\ 0&1&1&1&1&1&0&0 \\ 0&0&1&1&1&1&1&0 \\ 0&0&0&1&1&1&1&1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_7 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

**Comment.** The particular choice of matrix and vector has the effect that no ByteSub output equals the ByteSub input (or its complement).

**Example 138.** Invert $x^3+1$ in $\mathrm{GF}(2^8)$, constructed as in AES. [Example 135, again]

**Solution.** We use the extended Euclidean algorithm, and always reduce modulo $2$:

$$\begin{aligned} \boxed{x^8+x^4+x^3+x+1} &\equiv (x^5+x^2+x+1)\cdot\boxed{x^3+1}+x^2 \\ \boxed{x^3+1} &\equiv x\cdot\boxed{x^2}+1 \end{aligned}$$

Backtracking through this, we find that Bézout's identity takes the form

$$\begin{aligned} 1 \equiv 1\cdot\boxed{x^3+1}-x\cdot\boxed{x^2} &\equiv 1\cdot\boxed{x^3+1}-x\cdot\left(\boxed{x^8+x^4+x^3+x+1}-(x^5+x^2+x+1)\cdot\boxed{x^3+1}\right) \\ &\equiv (x^6+x^3+x^2+x+1)\cdot\boxed{x^3+1}+x\cdot\boxed{x^8+x^4+x^3+x+1}. \end{aligned}$$

Hence, $(x^3+1)^{-1}=x^6+x^3+x^2+x+1$ in $\mathrm{GF}(2^8)$.

**Example 139. (homework)**

(a) What happens to the byte $(0000\ 0101)_2$ during ByteSub?

(b) What happens to the byte $(0000\ 1001)_2$ during ByteSub?

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**Solution.**

(a) $(0000\ 0101)_2$ represents the polynomial $x^2+1$.
By the previous example, its inverse is $(x^2+1)^{-1} = x^6+x^4+x$ in $\mathrm{GF}(2^8)$, which is $\boldsymbol{c} = (0101\ 0010)_2$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{c}} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

[This is just the usual matrix-vector product modulo $2$. The highlighted columns are the ones which get added up during this matrix-vector product.]

Hence, the output of ByteSub is the byte $(0110\ 1011)_2$.

**Check with lookup tables.** Indeed, our computation matches $107 = (0110\ 1011)_2$ in the lookup table in our book (row $0$, column $(0101)_2 = 5$) or $(6\mathrm{B})_{16} = (0110\ 1011)_2$ on wikipedia (row $(0000)_2 = (0)_{16}$, column $(0101)_2 = (5)_{16}$).

(b) $(0000\ 1001)_2$ represents the polynomial $x^3+1$.
By the previous example, $(x^3+1)^{-1} = x^6+x^3+x^2+x+1$ in $\mathrm{GF}(2^8)$, which is $\boldsymbol{c} = (0100\ 1111)_2$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{c}} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, the output of ByteSub is the byte $(0000\ 0001)_2$.

**Check with lookup tables.** Indeed, our computation matches the value $1$ in the lookup table in our book (row $0$, column $(1001)_2 = 9$) or $(01)_{16}$ on wikipedia (row $(0000)_2 = (0)_{16}$, column $(1001)_2 = (9)_{16}$).

## Review: multiplicative order and primitive roots

**Definition 140.** The **multiplicative order** of an invertible residue $a$ modulo $n$ is the smallest positive integer $k$ such that $a^k \equiv 1 \pmod{n}$.

**Important note.** By Euler's theorem, the multiplicative order can be at most $\phi(n)$.

**Example 141.** What is the multiplicative order of $2 \pmod 7$?

**Solution.** $2^1 = 2$, $2^2 = 4$, $2^3 \equiv 1 \pmod 7$. Hence, the multiplicative order of $2 \pmod 7$ is $3$.

**Definition 142.** If the multiplicative order of an residue $a$ modulo $n$ equals $\phi(n)$ [in other words, the order is as large as possible], then $a$ is said to be **primitive root** modulo $n$.

A primitive root is also referred to as a **multiplicative generator** (because the products of $a$ and itself, that is, $1, a, a^2, a^3, \ldots$, produce all invertible residues).

**Example 143.** What is the multiplicative order of $3 \pmod 7$?

**Solution.** $3^1 = 3$, $3^2 \equiv 2$, $3^3 \equiv 6$, $3^4 \equiv 4$, $3^5 \equiv 5$, $3^6 \equiv 1$. Hence, the multiplicative order of $3 \pmod 7$ is $6$. This means that $3$ is a primitive root modulo $7$. Note how every (invertible) residue shows up as a power of $3$.