Example 103. In 12/2018, a new largest (proven) prime was found: $2^{82,589,933} - 1$.

https://www.mersenne.org/primes/?press=M82589933

This is a **Mersenne prime** (like the last 17 record primes). It has a bit over 24.8 million (decimal) digits (versus 23.2 for the previous record). The prime was found as part of GIMPS (Great Internet Mersenne Prime Search), which offers a $3,000 award for each new Mersenne prime discovered.

The EFF (Electronic Frontier Foundation) is offering $150,000 (donated anonymously for that specific purpose) for the discovery of the first prime with at least 100 million decimal digits.

https://www.eff.org/awards/coop

[Prizes of $50,000 and $100,000 for primes with 1 and 10 million digits have been claimed in 2000 and 2009.]

Extra excursion on Mersenne primes

Definition 104. A **Mersenne prime** is a prime of the form $2^n - 1$.

For instance. The first few Mersenne primes have exponents $2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, \ldots$ All of these exponents are primes (but not all primes work: for instance, $2^{11} - 1 = 23 \cdot 89$). See below.

Anecdote. Euler proved in 1772 that $2^{31} - 1$ is prime (then, and until 1867, the largest known prime).

"$2^{31} - 1$ is probably the greatest [Mersenne prime] that ever will be discovered; for as they are merely curious, without being useful, it is not likely that any person will attempt to find one beyond it." — P. Barlow, 1811

https://en.wikipedia.org/wiki/2,147,483,647

Mersenne primes give rise precisely to all even perfect numbers (numbers whose proper divisors sum to the number itself; for instance, 6 is perfect because $6 = 1 + 2 + 3$). Indeed, Euclid showed that, if $2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is perfect [$p = 2$: $2 \cdot 3 = 6$, $p = 3$: $4 \cdot 7 = 28 = 1 + 2 + 4 + 7 + 14$, $p = 5$: $16 \cdot 33 = 528$, ...]. It is not known whether odd perfect numbers exist.

Example 105. (geometric sum) Evaluate $1 + x + x^2 + \ldots + x^n$.

Solution. $(1 + x + x^2 + \ldots + x^n)(x - 1) = x^{n+1} - 1$, so that $1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1}$.

Geometric series. In particular, $\sum_{k=1}^{\infty} x^k = \lim_{n \to \infty} \frac{x^{n+1} - 1}{x - 1} = \frac{1}{1 - x}$, provided that $|x| < 1$.

Lemma 106. If $r \mid n$, then $x^r - 1 \mid x^n - 1$.

Proof. Write $n = rs$. It follows from $x^s - 1 = (x - 1)(1 + x + x^2 + \ldots + x^{s-1})$ that

$x^{rs} - 1 = (x^r - 1)(1 + x^r + x^{2r} + \ldots + x^{r(s-1)})$. □

Corollary 107. $2^n - 1$ can only be prime if n is prime.

Proof. It follows from the previous lemma that, if $n = rs$ is composite, then $2^n - 1$ is divisible by $2^r - 1$ (as well as $2^s - 1$). □

For instance. $2^6 - 1 = 63$ is divisible by both $2^2 - 1 = 3$ and $2^3 - 1 = 7$.

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