**Example 210. (Riemann hypothesis)** The **Riemann zeta function** $\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$ converges (for real $s$) if and only if $s > 1$.

The divergent series $\zeta(1)$ is the harmonic series, and $\zeta(p)$ is often called a $p$-series in Calculus II.

**Comment.** Euler achieved worldwide fame in 1734 by discovering and proving that $\zeta(2) = \frac{\pi^2}{6}$ (and similar formulas for $\zeta(4), \zeta(6), \ldots$).

For complex values of $s \neq 1$, there is a unique way to "analytically continue" this function. It is then "easy" to see that $\zeta(-2) = 0$, $\zeta(-4) = 0$, .... The **Riemann hypothesis** claims that all other zeroes of $\zeta(s)$ lie on the line $s = \frac{1}{2} + a\sqrt{-1}$ ($a \in \mathbb{R}$). A proof of this conjecture (checked for the first 10,000,000,000 zeroes) is worth $1,000,000.

http://www.claymath.org/millennium-problems/riemann-hypothesis

**The connection to primes.** Here's a vague indication that $\zeta(s)$ is intimately connected to prime numbers:

$$\begin{aligned}\zeta(s) &= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \ldots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \ldots\right)\left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \ldots\right)\cdots \\ &= \frac{1}{1 - 2^{-s}} \frac{1}{1 - 3^{-s}} \frac{1}{1 - 5^{-s}} \cdots \\ &= \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}\end{aligned}$$

This infinite product is called the Euler product for the zeta function. If the Riemann hypothesis was true, then we would be better able to estimate the number $\pi(x)$ of primes $p \leqslant x$.

More generally, certain statements about the zeta function can be translated to statements about primes. For instance, the (non-obvious!) fact that $\zeta(s)$ has no zeros for $\mathrm{Re}\, s = 1$ implies the prime number theorem.

http://www-users.math.umn.edu/~garrett/m/v/pnt.pdf

[ ... brief intro to elliptic curves: more next time! ...]

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