Review. $\mathrm{GF}(p^n)$ is "the" finite field with p^n elements.

Recall that, in the construction of $\mathrm{GF}(p^n)$, the polynomial $m(x)$ has to be such that it cannot be factored modulo p. We also say that $m(x)$ needs to be **irreducible** mod p.

For instance. The polynomial x^2+2x+1 can always be factored as $(x+1)^2$.

On the other hand. For the polynomials $m(x)=x^2+x+1$ things are more interesting:

- x^2+x+1 cannot be factored over $\mathbb{Q}$ because the roots $\frac{-1\pm\sqrt{-3}}{2}$ are not rational.
- However, $x^2+x+1\equiv(x+2)^2$ modulo 3, so it can be factored modulo 3.
- On the other hand, x^2+x+1 is irreducible modulo 2 (that is, it cannot be factored: the only linear factors are x and $x+1$, but x^2, $x(x+1)$ and $(x+1)^2$ are all different from x^2+x+1 modulo 2).

In general, it follows from the formula $\frac{-1\pm\sqrt{-3}}{2}$ for the roots that x^2+x+1 can be factored modulo a prime $p>2$ if and only if $\sqrt{-3}$ exists as a residue modulo p. In other words, if and only if -3 is a quadratic residue modulo p.

For instance. Modulo $p=7$, we have $-3\equiv 2^2$ and $\frac{1}{2}\equiv 4$, so that $\frac{-1\pm\sqrt{-3}}{2}\equiv 4\cdot(-1\pm 2)\equiv 2,4$. Indeed, we have the factorization $(x-2)(x-4)=x^2-6x+8\equiv x^2+x+1$ modulo 7.

Example 129. The polynomial x^3+x+1 is irreducible modulo 2, so we can use it to construct the finite field $\mathrm{GF}(2^3)$ with 8 elements.

(a) List all 8 elements.

(b) Reduce x^5+1 in $\mathrm{GF}(2^3)$.

(c) Multiply each element of $\mathrm{GF}(2^3)$ with x^2+x.

(d) What is the inverse of x^2+x in $\mathrm{GF}(2^3)$?

Solution.

(a) The elements are $0,1,x,x+1,x^2,x^2+1,x^2+x,x^2+x+1$.

[Note that $x^3=-x-1=x+1$ in $\mathrm{GF}(2^3)$. That means all polynomials of degree 3 and higher can be reduced to polynomials of degree less than 3. See next part.]

(b) We divide x^5+1 by x^3+x+1 (long division!) to find $x^5+1=(x^2-1)(x^3+x+1)+(-x^2+x+2)$. It follows that x^5+1 reduces to $-x^2+x+2\equiv x^2+x$ in $\mathrm{GF}(2^3)$.

Important. We can simplify things by performing the long division modulo 2. We then find $x^5+1\equiv(x^2+1)(x^3+x+1)+(x^2+x)$.

(c) We multiply the polynomials as usual, then reduce as in the previous part.

For instance, $(x^2+x)(x^2+x+1)\equiv x^4+x$ and, by long division, $x^4+x\equiv x(x^3+x+1)+x^2$, which reduces to just x^2 in $\mathrm{GF}(2^3)$.

$\times$	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
x^2+x	0	x^2+x	x^2+x+1	1	x^2+1	$x+1$	x	x^2

(d) We are looking for an element y such that $y(x^2+x)=1$ in $\mathrm{GF}(2^3)$. Looking at the table, we see that $y=x+1$ has that property. Hence, $(x^2+x)^{-1}=x+1$ in $\mathrm{GF}(2^3)$.

Important. To find the inverse, we essentially tried all possibilities. That's not sustainable. Instead, we can (and should!) proceed as we did for computing the inverse of residues modulo n. That is, we should use the Euclidean algorithm as indicated in the next examples. Here, this is just one step: modulo 2, we have $\boxed{x^3+x+1}\equiv(x+1)\cdot\boxed{x^2+x}+1$, so that $(x^2+x)^{-1}=x+1$ in $\mathrm{GF}(2^3)$.

Example 130.

(a) Apply the extended Euclidean algorithm to find the gcd of x^2+1 and x^4+x+1, and spell out Bezout's identity.

(b) Repeat the previous computation but always reduce all coefficients modulo 2.

(c) What is the inverse of x^2+1 in $\mathrm{GF}(2^4)$? Here, $\mathrm{GF}(2^4)$ is constructed using x^4+x+1.

Solution.

(a) We use the extended Euclidean algorithm:

$$\begin{array}{ll} \gcd(x^2+1, x^4+x+1) & \boxed{x^4+x+1} = (x^2-1)\cdot\boxed{x^2+1} + (x+2) \\ = \gcd(x+2, x^2+1) & \boxed{x^2+1} = (x-2)\cdot\boxed{x+2} + 5 \\ = 5 & \end{array}$$

Backtracking through this, we find that Bézout's identity takes the form

$$\begin{aligned} 5 &= 1\cdot\boxed{x^2+1} - (x-2)\cdot\boxed{x+2} = 1\cdot\boxed{x^2+1} - (x-2)\cdot\left(\boxed{x^4+x+1} - (x^2-1)\cdot\boxed{x^2+1}\right) \\ &= (x^3-2x^2-x+3)\cdot\boxed{x^2+1} - (x-2)\cdot\boxed{x^4+x+1} \end{aligned}$$

If we wanted to, we could divide both sides by 5.

(b) We repeat the exact same computation but reduce modulo 2 at each step:

$$\begin{aligned} \boxed{x^4+x+1} &\equiv (x^2+1)\cdot\boxed{x^2+1} + x \\ \boxed{x^2+1} &\equiv = x\cdot\boxed{x} + 1 \end{aligned}$$

Backtracking through this, we find that Bézout's identity takes the form

$$\begin{aligned} 1 &= 1\cdot\boxed{x^2+1} + x\cdot\boxed{x} = 1\cdot\boxed{x^2+1} + x\cdot\left(\boxed{x^4+x+1} + (x^2+1)\cdot\boxed{x^2+1}\right) \\ &= (x^3+x+1)\cdot\boxed{x^2+1} + x\cdot\boxed{x^4+x+1} \end{aligned}$$

(c) We can now read off that $(x^2+1)^{-1} = x^3+x+1$ in $\mathrm{GF}(2^4)$.

Example 131. (HW) Find the inverses of x^2+1 and x^3+1 in $\mathrm{GF}(2^8)$, constructed as in AES.

Solution. Recall that for AES, $\mathrm{GF}(2^8)$ is constructed using $x^8+x^4+x^3+x+1$.

(a) We use the extended Euclidean algorithm for polynomials, and reduce all coefficients modulo 2:

$$\boxed{x^8+x^4+x^3+x+1} \equiv (x^6+x^4+x)\cdot\boxed{x^2+1} + 1$$

Hence, $(x^2+1)^{-1} = x^6+x^4+x$ in $\mathrm{GF}(2^8)$.

(b) We use the extended Euclidean algorithm, and always reduce modulo 2:

$$\begin{aligned} \boxed{x^8+x^4+x^3+x+1} &\equiv (x^5+x^2+x+1)\cdot\boxed{x^3+1} + x^2 \\ \boxed{x^3+1} &\equiv x\cdot\boxed{x^2} + 1 \end{aligned}$$

Backtracking through this, we find that Bézout's identity takes the form

$$\begin{aligned} 1 &\equiv 1\cdot\boxed{x^3+1} - x\cdot\boxed{x^2} \equiv 1\cdot\boxed{x^3+1} - x\cdot\left(\boxed{x^8+x^4+x^3+x+1} - (x^5+x^2+x+1)\cdot\boxed{x^3+1}\right) \\ &\equiv (x^6+x^3+x^2+x+1)\cdot\boxed{x^3+1} + x\cdot\boxed{x^8+x^4+x^3+x+1}. \end{aligned}$$

Hence, $(x^3+1)^{-1} = x^6+x^3+x^2+x+1$ in $\mathrm{GF}(2^8)$.

Armin Straub
straub@southalabama.edu