Example 135. (cont'd)

Solution. $\int_C f(x,y)\mathrm{d}s = \int_{C_1} f(x,y)\mathrm{d}s + \int_{C_2} f(x,y)\mathrm{d}s = \int_0^\pi f(2\cos(t), 2\sin(t))2\mathrm{d}t + \int_{-2}^2 f(t,0)\mathrm{d}t$

$$\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_{C_1} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} + \int_{C_2} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_0^\pi \begin{bmatrix} f \\ g \end{bmatrix}\cdot\begin{bmatrix} -2\sin(t) \\ 2\cos(t) \end{bmatrix}\mathrm{d}t + \int_{-2}^2 \begin{bmatrix} f \\ g \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathrm{d}t$$
$$= \int_0^\pi \left[-2\sin(t) f(2\cos(t), 2\sin(t)) + 2\cos(t) g(2\cos(t), 2\sin(t))\right]\mathrm{d}t + \int_{-2}^2 f(t,0)\mathrm{d}t$$

Example 136. If $f(x,y)$, then $\boldsymbol{F} = \nabla f = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix}$ is a vector field. We say that $\boldsymbol{F}$ is a **gradient field** and f is a **potential function** for $\boldsymbol{F}$. E.g., f gravitational potential, $\boldsymbol{F}$ gravitational field.

Theorem 137. (Fundamental Theorem of Line Integrals) Let C be a curve from A to B.

$$\int_C \nabla f\cdot\mathrm{d}\boldsymbol{r} = f(B) - f(A)$$

Fine print: naturally, f needs to be continuously differentiable on a domain containing the entire curve C.

This is a generalization of the Fundamental Theorem of Calculus: $\int_a^b f'(t)\mathrm{d}t = f(b) - f(a)$.

A vector field $\boldsymbol{F}$ is **conservative** (on a region D) if for any curve C (in D) the line integral $\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$ is **path independent** (i.e. it only depends on the start and end point of C).

The Fundamental Theorem of Line Integrals says that gradient fields are conservative. The following result asserts that the converse is true:

Theorem 138. $\boldsymbol{F}$ is a conservative field if and only if $\boldsymbol{F}$ is a gradient field.

Fine print: the statements are about an open region D on which $\boldsymbol{F}$ is continuous.

Example 139. Let C be the curve from $(0,0,0)$ to $(1,1,2)$ parametrized by $\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j} + 2t\boldsymbol{k}$, $t\in[0,1]$, and let L be the line segment from $(0,0,0)$ to $(1,1,2)$.

Let $f(x,y,z) = xyz$, and $\boldsymbol{F} = \nabla f = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$. Evaluate $\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$ and $\int_L \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$.

Solution. (with Fundamental Theorem of Line Integrals) Both curves are from $A = (0,0,0)$ to $B = (1,1,2)$. Since $\boldsymbol{F} = \nabla f$, the line integral over $\boldsymbol{F}$ is path independent and, in both cases,

$$\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_L \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = f(B) - f(A) = \left[xyz\right]_{(0,0,0)}^{(1,1,2)} = 1\cdot1\cdot2 - 0\cdot0\cdot0 = 2.$$

Solution. (without Fundamental Theorem of Line Integrals) Do it! For instance,

$$\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_0^1 \boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)\mathrm{d}t = \int_0^1 \begin{bmatrix} t^2\cdot 2t \\ t\cdot 2t \\ t\cdot t^2 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 2t \\ 2 \end{bmatrix}\mathrm{d}t = \int_0^1 (2t^3 + 4t^3 + 2t^3)\mathrm{d}t = \left[2t^4\right]_0^1 = 2.$$

Example 140. Redo Example 139 with $\boldsymbol{F} = xz\boldsymbol{i} - xy\boldsymbol{k}$.

Solution. Do it! Your final answers should be $\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \frac{1}{6}$ and $\int_L \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = 0$.

In particular, we conclude that $\boldsymbol{F}$ cannot possibly be a gradient field (i.e. there is no "antiderivative" [**potential** is the professional word] f such that $\boldsymbol{F} = \nabla f$).

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