**Line integrals** (of vector fields): $\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t))\cdot \boldsymbol{r}'(t)\mathrm{d}t$

If $\boldsymbol{F} = \begin{bmatrix} f \\ g \end{bmatrix}$, then $\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_a^b \begin{bmatrix} f \\ g \end{bmatrix}\cdot \boldsymbol{r}'(t)\mathrm{d}t = \int_C \begin{bmatrix} f \\ g \end{bmatrix}\cdot \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \int_C f\mathrm{d}x + g\mathrm{d}y$.

- Here, $\boldsymbol{F}(\boldsymbol{r}(t)) = \boldsymbol{F}(x(t), y(t))$ or $\boldsymbol{F}(\boldsymbol{r}(t)) = \boldsymbol{F}(x(t), y(t), z(t))$.
- Note that, for short, we are writing $\int_C f\mathrm{d}x + g\mathrm{d}y = \int_C f(x,y)\mathrm{d}x + \int_C g(x,y)\mathrm{d}y$.
- Functions $\boldsymbol{F}(x,y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ or $\boldsymbol{F}(x,y,z) = \begin{bmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{bmatrix}$ are called **vector fields**. More later!
- This line integral is also sometimes written as $\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_C \boldsymbol{F}\cdot \boldsymbol{T}\mathrm{d}s$, where $\boldsymbol{T} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ is a unit vector tangent to the curve $C$ (at each point).
- We will briefly see later that $\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$ can be interpreted as the amount of work required to move an object along the curve $C$ through the force field $\boldsymbol{F}$.

**Example 133.** Evaluate $\int_C \mathrm{d}y$, $\int_C x\mathrm{d}y$ and $\int_C \begin{bmatrix} y \\ x \end{bmatrix}\cdot \mathrm{d}\boldsymbol{r} = \int_C y\mathrm{d}x + x\mathrm{d}y$ where $C$ is the straight-line segment from $(0,1)$ to $(1,0)$. (compare with example from two classes ago)

**Solution.** We again use $\boldsymbol{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t \\ 1-t \end{bmatrix}$, $t\in[0,1]$. (You can pick any other parametrization of the line segment as long as it starts at $(0,1)$ and ends at $(1,0)$.)

$\int_C \mathrm{d}y = \int_0^1 y'(t)\mathrm{d}t = \int_0^1 (-1)\mathrm{d}t = -1$ (This is just the total change in $y$!)

$\int_C x\mathrm{d}y = \int_0^1 t(-1)\mathrm{d}t = \left[-\frac{t^2}{2}\right]_0^1 = -\frac{1}{2}$

$\int_C \begin{bmatrix} y \\ x \end{bmatrix}\cdot \mathrm{d}\boldsymbol{r} = \int_0^1 \begin{bmatrix} y(t) \\ x(t) \end{bmatrix}\cdot \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}\mathrm{d}t = \int_0^1 \begin{bmatrix} 1-t \\ t \end{bmatrix}\cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}\mathrm{d}t = \int_0^1 (1-2t)\mathrm{d}t = \left[t-t^2\right]_0^1 = 0$

**Alternative.** $\int_C \begin{bmatrix} y \\ x \end{bmatrix}\cdot \mathrm{d}\boldsymbol{r} = \int_C y\mathrm{d}x + x\mathrm{d}y = \int_0^1 y(t)x'(t)\mathrm{d}t + \int_0^1 x(t)y'(t)\mathrm{d}t = \ldots = 0$

If $-C$ denotes the same curve as $C$ but traversed in opposite direction, then

$$\int_{-C} f(x,y)\mathrm{d}y = -\int_C f(x,y)\mathrm{d}y \text{ while } \int_{-C} f(x,y)\mathrm{d}s = \int_C f(x,y)\mathrm{d}s.$$

The first part is a version of $\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$. For the integral with $\mathrm{d}s$ there is no change of the sign because $\mathrm{d}s$ is a little length (always $\geqslant 0$). On the other hand, $\mathrm{d}y$ is a change in $y$-coordinate and can be either positive or negative (and going in the opposite direction flips that sign!).

**Example 134. (HW)** Spell out (i.e. express as ordinary integrals) the line integrals $\int_C f(x,y)\mathrm{d}s$, $\int_C f(x,y)\mathrm{d}x$ and $\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$, with $\boldsymbol{F} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$, where $C$ is the boundary of $x^2+y^2 \leqslant 4$, $y\geqslant 0$, starting and ending at $(2,0)$ and traversed in counterclockwise direction.

**Solution.** Make a sketch of $C$! We break $C$ into $C_1$ and $C_2$, with $C_1$ parametrized by $\boldsymbol{r}(t) = \begin{bmatrix} 2\cos(t) \\ 2\sin(t) \end{bmatrix}$, from $t=0$ to $t=\pi$, and $C_2$ parametrized by $\boldsymbol{r}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}$, from $t=-2$ to $t=2$. Then, for instance,

$$\int_C f(x,y)\mathrm{d}x = \int_{C_1} f(x,y)\mathrm{d}x + \int_{C_2} f(x,y)\mathrm{d}x = \int_0^\pi f(2\cos(t), 2\sin(t))(-2\sin(t))\mathrm{d}t + \int_{-2}^2 f(t,0)\mathrm{d}t.$$


Armin Straub
straub@southalabama.edu