Example 104. Find all local maxima, local minima and saddle points of the function $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$.

Solution. To find the critical points, we need to solve the two equations $f_x = 12x - 6x^2 + 6y = 0$ and $f_y = 6y + 6x = 0$ for the two unknowns x, y.

[A general strategy is to solve one equation for one variable (in terms of the other), and substitute that in the other equation. Then we have a single equation in a single variable, which we can solve.]

Here, the second equation simplifies to $y = -x$. Substituting that in the first equation, we get $12x - 6x^2 - 6x = 6x - 6x^2 = 6x(1-x)$. Hence, $x = 0$ or $x = 1$.

If $x = 0$ then $y = -x = 0$, and we get the point $(0,0)$. If $x = 1$ then $y = -x = -1$, and we get the point $(1,-1)$.

In conclusion, the critical points are $(0,0)$, $(1,-1)$.

$\left[f_{xx}f_{yy} - f_{xy}^2\right]_{(0,0)} = \left[(12-12x)\cdot 6 - 6^2\right]_{(0,0)} = 36 > 0$ and $f_{xx} = 12 > 0$. Hence, $(0,0)$ is a local minimum.

$\left[f_{xx}f_{yy} - f_{xy}^2\right]_{(1,-1)} = \left[(12-12x)\cdot 6 - 6^2\right]_{(1,-1)} = -36 < 0$. Hence, $(1,-1)$ is a saddle point.

Example 105. (a simple saddle point) Find the local extreme values of $f(x,y) = x^2 - y^2$.

Solution. $f_x = 2x = 0$ gives $x = 0$, and $f_y = -2y = 0$ gives $y = 0$. Only critical points: $(0,0)$

$f_{xx}f_{yy} - f_{xy}^2 = 2\cdot(-2) - 0 = -4 < 0$. Hence, $(0,0)$ is a saddle point.

Comment. Make a sketch of the graph of $f(x,y)$ restricted to the xz-plane (this restriction has a minimum at the origin) and another sketch restricted to the yz-plane (this restriction has a maximum at the origin). See Figure 13.44 in our book for a 3-dimensional sketch. It looks like the **saddle** for horseback riding!

Lagrange multipliers

(Lagrange multipliers) To find local extrema of $f(x,y,z)$ subject to the constraint $g(x, y, z) = 0$, find values x, y, z, λ such that

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x,y,z) = 0.$$

Fine print: of course, f and g need to be differentiable. We also need $\nabla g \neq 0$ when $g(x,y,z) = 0$.

In other words, at such a local extremum, we should have that ∇f and ∇g point in the same direction!

Why? The derivative of f in directions $\boldsymbol{u}$ allowed by the constraint should be zero. Since these directional derivatives are $\nabla f \cdot \boldsymbol{u}$, this means that ∇f should be orthogonal to $g(x,y,z) = 0$. That in turn means that ∇f and ∇g point in the same direction. [If you can reproduce this argument, you really understand the gradient!]

Let us redo Example 62 using this new machinery.

Example 106. Find the point R on the plane $2x + 2y - z = 3$ closest to $S = (2,3,0)$.

Solution. First off, let us write this as a minimization problem. We wish to minimize $f(x,y,z) = (x-2)^2 + (y-3)^2 - z^2$ subject to the constraint $g(x,y,z) = 2x + 2y - z - 3 = 0$.

$\nabla f = \begin{bmatrix} 2(x-2) \\ 2(y-3) \\ 2z \end{bmatrix}$, $\nabla g = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$. We need to solve the four equations $2(x-2) = 2\lambda$, $2(y-3) = 2\lambda$, $2z = -\lambda$, $2x + 2y - z - 3 = 0$ for the four unknowns λ, x, y, z.

Try to do it! (The equations are linear, so exactly the kind you learn to solve systematically in linear algebra.)

In the end, we find $\lambda = -\frac{14}{9}$, $x = \frac{4}{9}$, $y = \frac{13}{9}$, $z = \frac{7}{9}$. Hence, $R = \frac{1}{9}\begin{bmatrix} 4 \\ 13 \\ 7 \end{bmatrix}$, as in our previous solution.

Armin Straub
straub@southalabama.edu