**Example 66.** If $P(t)=\begin{bmatrix}\cos(t)\\ \sin(t)\end{bmatrix}$, then $P'(t)=\begin{bmatrix}-\sin(t)\\ \cos(t)\end{bmatrix}$. This is the tangent vector to the curve (here, the unit circle) parametrized by $P(t)$.

Sketch the situation in the cases $t=0$, $t=\frac{\pi}{4}$, $t=\frac{\pi}{2}$.

If $P(t)$ is the position of a particle at time $t$, then:

- $\boldsymbol{v}(t)=P'(t)=\frac{\mathrm{d}P}{\mathrm{d}t}$ is the **velocity**,

  $|\boldsymbol{v}(t)|$ is the **speed** (scalar!),

  $\frac{\boldsymbol{v}(t)}{|\boldsymbol{v}(t)|}$ is the **direction of motion** (a unit vector),
- $\boldsymbol{a}(t)=P''(t)=\frac{\mathrm{d}^2P}{\mathrm{d}t^2}$ is the **acceleration**.

**Example 67.** Let $P(t)=\begin{bmatrix}\cos(t)\\ \sin(t)\end{bmatrix}$ be the position of a particle at time $t$. Find its velocity, speed and acceleration at time $t$.

**Solution.** The velocity is $\boldsymbol{v}(t)=P'(t)=\begin{bmatrix}-\sin(t)\\ \cos(t)\end{bmatrix}$.

The speed is $|\boldsymbol{v}(t)|=\sqrt{(-\sin t)^2+(\cos t)^2}=1$. (Note that our particle is traveling at constant speed.)

The acceleration is $\boldsymbol{a}(t)=P''(t)=\begin{bmatrix}-\cos(t)\\ -\sin(t)\end{bmatrix}$.

Sketch the situation for $t=0$ and $t=\pi/4$!

**Note.** Observe that the acceleration is always perpendicular to the velocity. We will show below that this happens always when the speed is constant.

Vector functions satisfy the expected rules for differentiating.

For instance (very boring, and obvious): $\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{v}(t)+\boldsymbol{w}(t)]=\boldsymbol{v}'(t)+\boldsymbol{w}'(t)$

The following product rules are not surprising either: [as frequently done, we write $\boldsymbol{v}$ for $\boldsymbol{v}(t)$]

$$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{v}\cdot\boldsymbol{w}]=\boldsymbol{v}'\cdot\boldsymbol{w}+\boldsymbol{v}\cdot\boldsymbol{w}'$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{v}\times\boldsymbol{w}]=\boldsymbol{v}'\times\boldsymbol{w}+\boldsymbol{v}\times\boldsymbol{w}'$$

Why do these hold? For instance, $\boldsymbol{v}\cdot\boldsymbol{w}=v_1w_1+v_2w_2+v_3w_3$ and we can apply the usual product rule to each of the terms $v_iw_i$.

**Example 68.** Why is the following true? If a particle moves at constant speed, then its velocity and acceleration vectors are always perpendicular.

**Solution.** Constant speed means that $|\boldsymbol{v}|=c$ for some constant $c$. Taking the derivative of $\boldsymbol{v}\cdot\boldsymbol{v}=|\boldsymbol{v}|^2=c^2$, we get $\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{v}\cdot\boldsymbol{v}]=\boldsymbol{v}'\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{v}'=2\boldsymbol{v}\cdot\boldsymbol{a}=0$. In other words, $\boldsymbol{v}$ and $\boldsymbol{a}$ are perpendicular.

In other words, if the speed is constant, then the tangential part of the acceleration is zero.

We will discuss this further next time.

Armin Straub
straub@southalabama.edu