In summary, the **cross product** of $\boldsymbol{v}$ and $\boldsymbol{w}$ is:

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\boldsymbol{v}} \times \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}_{\boldsymbol{w}} = \begin{bmatrix} v_2 w_3 - w_2 v_3 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

$$= \underbrace{|\boldsymbol{v}|\,|\boldsymbol{w}|\sin\theta}_{\text{length}} \quad \underbrace{\boldsymbol{n}}_{\substack{\text{direction} \\ \text{(unit vector orthogonal to } \boldsymbol{v} \text{ and } \boldsymbol{w})}}$$

- How can we see that the "simple" formula for $\boldsymbol{v} \times \boldsymbol{w}$ is correct?

  For instance, $v_2 w_3 - w_2 v_3$ is the $\boldsymbol{i}$ component of $\boldsymbol{v} \times \boldsymbol{w}$. "How can we get an $\boldsymbol{i}$?" Well, $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$ and $\boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$. In the first case, we take the $\boldsymbol{j}$ component of $\boldsymbol{v}$ $(= v_2)$ and the $\boldsymbol{k}$ component of $\boldsymbol{w}$ $(= w_3)$. Together, $v_2 w_3$. (Likewise, $\boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$ gives us $-w_2 v_3$.)
- Those of you, who know $3 \times 3$ determinants, might find the following mnemonic useful:

  $\langle v_1, v_2, v_3 \rangle \times \langle w_1, w_2, w_3 \rangle = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$ [Just ignore for now, if you don't know determinants.]

**Example 36.** Compute $(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) \times (\boldsymbol{i} - 2\boldsymbol{k})$.

**Solution.** $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -4-0 \\ 3-(-2) \\ 0-2 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix}$

As earlier, let us check our answer: $\begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -4 + 10 - 6 = 0$ and $\begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = -4 + 0 + 4 = 0$

$|\boldsymbol{v} \times \boldsymbol{w}|$ is the **area of the parallelogram** with sides $\boldsymbol{v}$ and $\boldsymbol{w}$.

**Why?** Make a sketch! (See, for instance, Figure 11.30 in the book.) The area of a parallelogram is "base times height". Take, for instance, $\boldsymbol{v}$ as your base, so the base has length $|\boldsymbol{v}|$. By simple trigonometry, the height is $|\boldsymbol{w}|\sin\theta$. Hence, the area is $|\boldsymbol{v}|\,|\boldsymbol{w}|\sin\theta = |\boldsymbol{v} \times \boldsymbol{w}|$.

**Example 37.** Consider the triangle with vertices $P = (1,1,1)$, $Q = (2,1,3)$ and $R = (3,-1,1)$.

(a) Find the area of the triangle.

(b) Find a unit vector perpendicular to the plane $PQR$.

**Solution.**

(a) Let $\boldsymbol{v} = \overrightarrow{PQ}$ and $\boldsymbol{w} = \overrightarrow{PR}$ be two sides of our triangle (you can pick other sides, no problem!).
Then the area of the triangle is $\frac{1}{2}|\boldsymbol{v} \times \boldsymbol{w}|$. Here are the computations:

$$\boldsymbol{v} = \begin{bmatrix} 2-1 \\ 1-1 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \quad \boldsymbol{v} \times \boldsymbol{w} = \begin{bmatrix} 0-(-4) \\ 4-0 \\ -2-0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}.$$

Hence, the area is $\frac{1}{2}|\boldsymbol{v} \times \boldsymbol{w}| = \frac{1}{2}\sqrt{16+16+4} = 3$.

(b) The vector $\boldsymbol{v} \times \boldsymbol{w}$ is perpendicular to the plane $PQR$. We just have to scale it to a unit vector:

$$\frac{\boldsymbol{v} \times \boldsymbol{w}}{|\boldsymbol{v} \times \boldsymbol{w}|} = \frac{1}{6}\begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

Armin Straub
straub@southalabama.edu