Sketch of Lecture 10

Thu, 1/28/2016

The **cross product** (or **vector product**) is $\boldsymbol{v}\times\boldsymbol{w} = \underbrace{|\boldsymbol{v}|\,|\boldsymbol{w}|\sin\theta}_{\text{length}}\ \underbrace{\boldsymbol{n}}_{\text{direction}}$. [This is a vector!]

- Here, $\theta\in[0,\pi]$ is again the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$, and
- $\boldsymbol{n}$ is the unit vector orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$, chosen right-handedly.
- This product only makes sense in three dimensions!!

Contrast this with $\boldsymbol{v}\cdot\boldsymbol{w} = |\boldsymbol{v}|\,|\boldsymbol{w}|\cos\theta$. (Recall that the dot product makes sense in any dimension.)

Example 33. Determine $\boldsymbol{i}\times\boldsymbol{j}$ and $\boldsymbol{j}\times\boldsymbol{i}$.

Solution. The only vectors orthogonal to both $\boldsymbol{i}$ and $\boldsymbol{j}$ (in other words, vectors orthogonal to the xy-plane) are multiples $\boldsymbol{k}$ (vectors which in standard position are on the z-axis). The only such vectors of length 1 are $\boldsymbol{k}$ and $-\boldsymbol{k}$. To determine which of the two is the right one, we have to use the right-hand rule: use thumb for the first factor, index finger for second factor, and your middle finger will indicate the direction of the cross product. Doing so, we find $\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k}$ and $\boldsymbol{j}\times\boldsymbol{i}=-\boldsymbol{k}$.

$\boldsymbol{v}\times\boldsymbol{w} = -\boldsymbol{w}\times\boldsymbol{v}$ (This property is called **anticommutativity**.)

Example 34. Determine $\boldsymbol{i}\times\boldsymbol{i}$.

Solution. The angle between $\boldsymbol{i}$ and $\boldsymbol{i}$ is $\theta=0$, and so $\sin\theta=0$. Hence, $\boldsymbol{i}\times\boldsymbol{i}=\boldsymbol{0}=\langle 0,0,0\rangle$.
This is the **zero vector** (in 3D)! The zero vector is the only vector with length 0 and the only vector that has no direction.

$\boldsymbol{v}\times\boldsymbol{w}=\boldsymbol{0}$ if and only if $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel.

Contrast that with the dot product: $\boldsymbol{v}\cdot\boldsymbol{w}=0$ if and only if $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular.

We have to be very careful with the laws for the cross product! [Just saw that it is not quite commutative.]

- The cross product is not associative: $(\boldsymbol{u}\times\boldsymbol{v})\times\boldsymbol{w}\neq\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})$ (except, by "accidence")
- However, it is distributive: $\boldsymbol{u}\times(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{u}\times\boldsymbol{v}+\boldsymbol{u}\times\boldsymbol{w}$ (and the other way around)

Example 35. Using distributivity, compute $(\boldsymbol{i}+2\boldsymbol{j})\times(3\boldsymbol{i}-\boldsymbol{k})$.

Solution. $(\boldsymbol{i}+2\boldsymbol{j})\times(3\boldsymbol{i}-\boldsymbol{k}) = 3\underbrace{\boldsymbol{i}\times\boldsymbol{i}}_{\boldsymbol{0}} - \underbrace{\boldsymbol{i}\times\boldsymbol{k}}_{-\boldsymbol{j}} + 6\underbrace{\boldsymbol{j}\times\boldsymbol{i}}_{-\boldsymbol{k}} - 2\underbrace{\boldsymbol{j}\times\boldsymbol{k}}_{\boldsymbol{i}} = -2\boldsymbol{i}+\boldsymbol{j}-6\boldsymbol{k} = \langle -2,1,-6\rangle$

Important! How can we verify our computation?
A hugely important part of the cross product is that the resulting vector has to be orthogonal to the two original vectors. This is easy to check! Indeed, $\langle -2,1,-6\rangle\cdot\langle 1,2,0\rangle=0$ and $\langle -2,1,-6\rangle\cdot\langle 3,0,-1\rangle=0$.

We can now find a general formula for $\boldsymbol{v}\times\boldsymbol{w}$: start with $(v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k})\times(w_1\boldsymbol{i}+w_2\boldsymbol{j}+w_3\boldsymbol{k})$, multiply out and simplify! Here is what we get in the end:

$$\begin{bmatrix} v_1\\ v_2\\ v_3\end{bmatrix}\times\begin{bmatrix} w_1\\ w_2\\ w_3\end{bmatrix}=\begin{bmatrix} v_2w_3-w_2v_3\\ v_3w_1-v_1w_3\\ v_1w_2-v_2w_1\end{bmatrix}$$

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