Example 11. (continued) $\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x =$

Solution. Continuing from last time,

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \int_{\pi/2}^0 \sqrt{1-\cos^2(\theta)}\,(-\sin(\theta))\mathrm{d}\theta = \int_0^{\pi/2} \sqrt{1-\cos^2(\theta)}\,\sin(\theta)\mathrm{d}\theta.$$

Observe that $\sqrt{1-\cos^2(\theta)} = \sqrt{\sin^2(\theta)} = \sin(\theta)$ (the last step is valid here because $\sin(\theta) \geqslant 0$ for $\theta \in [0, \pi/2]$; otherwise, we would have to worry about the sign in $\sqrt{\sin^2(\theta)} = \pm\sin(\theta)$). Therefore, our integral is

$$\int_0^{\pi/2} \sqrt{1-\cos^2(\theta)}\,\sin(\theta)\mathrm{d}\theta = \int_0^{\pi/2} \sin^2(\theta)\mathrm{d}\theta = \int_0^{\pi/2} \frac{1-\cos(2\theta)}{2}\mathrm{d}\theta = \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^{\pi/2} = \frac{\pi}{4}.$$

(As a homework and review, evaluate $\int_0^{\pi/2} \sin^2(\theta)\mathrm{d}\theta$ via integration by parts; then use $\cos^2 = 1 - \sin^2$.)

Parametric curves

For instance, the motion of a particle in the xy-plane can be described by

$$x = f(t), \quad y = g(t), \quad t \in [a, b],$$

where t is the parameter (in this case, e.g., time). The trajectory of the particle is what we call a **parametric curve**.

This parametric curve has **arc length** $L = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$

Can you justify this formula with a sketch? Note that $\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}$.

Example 12. $x = \cos(t)$, $y = \sin(t)$ with $t \in [0, 2\pi]$.

(a) Describe the parametric curve.

(b) What is the arclength of the curve?

Solution.

(a) This is a circle of radius 1 around the origin. The curve starts at $(1, 0)$ (for $t = 0$) and returns to that same point (for $t = 2\pi$).

(b) $L = \int_0^{2\pi} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\,\mathrm{d}t = \int_0^{2\pi} 1\,\mathrm{d}t = 2\pi$

Similarly, the motion of a particle in space can be described by adding a third coordinate

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in [a, b].$$

This parametric curve has **arc length** $L = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$

Example 13. What is the arclength of the curve $x = \cos(t)$, $y = \sin(t)$, $z = t$ with $t \in [0, 2\pi]$?

Our next goal is to start working with coordinates in space carefully. This is just a motivational example that some things generalize to higher dimensions very pleasantly.

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