Midterm #2

MATH 126 — Calculus II
Friday, Nov 13

Please print your name:

Problem 1. Using the integral test, determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ converges.

Solution. By the integral test, the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ converges if and only if the integral $\int_2^{\infty} \frac{\mathrm{d}x}{x \log x}$ converges.

First, however, we should verify that the integral test indeed applies: the function $\frac{1}{x \log x}$ is obviously positive and continuous for $x \geqslant 2$. It is also decreasing, because $x \log x$ clearly increases.

Upon substituting $u = \log x$, we find that

$$\int_2^{\infty} \frac{\mathrm{d}x}{x \log x} = \int_{\log(2)}^{\infty} \frac{\mathrm{d}u}{u} = [\log |u|]_{\log(2)}^{\infty}$$

diverges because $\lim_{u \to \infty} \log |u| = \infty$. Therefore, the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges. □

Problem 2. Determine the following limits.

(a) $\lim_{n \to \infty} \frac{5^n + 3^n}{4^n - 1} =$ ∞

(b) $\lim_{n \to \infty} \frac{7n^2 - 8n}{2n^2 + 3} =$ $\frac{7}{2}$

(c) $\lim_{n \to \infty} \sqrt{\frac{3 + 2n^2}{1 + n + n^2}} =$ $\sqrt{2}$

(d) $\lim_{n \to \infty} \cos\left(\frac{n}{n^2 + 1}\right) =$ $\cos(0) = 1$

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Problem 3. Write down the geometric series. Under which condition does it converge, and what does it converge to?

Solution. The geometric series $\sum_{n=0}^{\infty} x^n$ converges if and only if $|x| < 1$. In that case, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. □

Problem 4. Under which condition does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

Solution. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. □

Problem 5. Determine whether the following series converge or diverge. Make sure to indicate a reason!

(a) $\sum_{n=2}^{\infty} \frac{1-\log(n)}{1+\log(n)}$

☐ series converges ☐ series diverges

Indicate a reason:

The series diverges because $\frac{1-\log(n)}{1+\log(n)} \to -1 \neq 0$ as $n \to \infty$.

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$

☐ series converges ☐ series diverges

Indicate a reason:

The series converges by limit comparison with the converging p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Indeed, if $a_n = \frac{n+1}{n^3+1}$ and $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

Hence, $\sum_{n=1}^{\infty} a_n$ converges because $\sum_{n=1}^{\infty} b_n$ does.

(c) $\sum_{n=2}^{\infty} \frac{7^n}{n^2 4^n}$

☐ series converges ☐ series diverges

Indicate a reason:

The series diverges because $\frac{7^n}{n^2 4^n} \to \infty \neq 0$ as $n \to \infty$.

(d) $\sum_{n=2}^{\infty} \frac{n+\sqrt{n}+7}{3n^2+1}$

☐ series converges ☐ series diverges

Indicate a reason:

The series diverges by limit comparison with the diverging harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Indeed, if $a_n = \frac{n+\sqrt{n}+7}{3n^2+1}$ and $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{3}$.

Hence, $\sum_{n=2}^{\infty} a_n$ diverges because $\sum_{n=2}^{\infty} b_n$ does.

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Problem 6. Consider the power series $\sum_{n=1}^{\infty} \frac{n}{5^n}(x+1)^n$

(a) Determine the radius of convergence R.

(b) Let $f(x) = \sum_{n=1}^{\infty} \frac{n}{5^n}(x+1)^n$ for x such that $|x+1| < R$. Write down a series for $f'(x)$.

Solution.

(a) We apply the ratio test with $a_n = \frac{n}{5^n}(x+1)^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+1)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n(x+1)^n}\right| = \frac{1}{5}|x+1|\frac{n+1}{n} \to \frac{1}{5}|x+1| \text{ as } n \to \infty$$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{n}{5^n}(x+1)^n$ converges if $\frac{1}{5}|x+1| < 1$ or, equivalently, $|x+1| < 5$.

The radius of convergence therefore is 5.

(b) $f'(x) = \sum_{n=1}^{\infty} \frac{n}{5^n} n(x+1)^{n-1} = \sum_{n=1}^{\infty} \frac{n^2}{5^n}(x+1)^{n-1}$ □

Problem 7. For which values of x does $\sum_{n=1}^{\infty} \frac{x^n+1}{2^n}$ converge? Evaluate the series (as a function of x) for these values.

Solution.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{x^n+1}{2^n} &= \sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n - 1 + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - 1 \\ [\text{if } |x/2| < 1] \quad &= \frac{1}{1-\frac{x}{2}} - 1 + \frac{1}{1-\frac{1}{2}} - 1 \\ &= \frac{2}{2-x} \end{aligned}$$

In particular, the series converges provided that $|x/2| < 1$, or, equivalently, $|x| < 2$. □

Problem 8. (Bonus!) What is the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$? [We don't have the tools to evaluate this series, but you might remember from class.]

Solution. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ □

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