Polar coordinates

The **polar coordinates** (r, θ) represent the point $(x, y) = r(\cos\theta, \sin\theta)$.

Often, θ is taken from $[0, 2\pi)$ (but $(-\pi, \pi]$ is another popular choice), and, usually, $r \geqslant 0$.

Example 208. Which point (in cartesian coordinates) has polar coordinates $r = 2$, $\theta = \frac{\pi}{6}$?

Solution. $(x, y) = r(\cos\theta, \sin\theta) = 2\left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) = \left(\sqrt{3}, 1\right)$

[Draw a right triangle with angle $\frac{\pi}{6} = 30°$ to find $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$.]

Note. The polar coordinates $r = 2$, $\theta = \frac{\pi}{6} + 2\pi$ correspond to the same point $\left(\sqrt{3}, 1\right)$. Polar coordinates are not quite unique.

Note. Sometimes, we permit negative r. For instance, the polar coordinates $r = -2$, $\theta = \frac{\pi}{6} + \pi$ also describe the point $\left(\sqrt{3}, 1\right)$.

How to calculate the polar coordinates (r, θ) for (x, y)?

By Pythagoras, $r = \sqrt{x^2 + y^2}$, and the angle is $\theta = \operatorname{atan2}(y, x) \in (-\pi, \pi]$.

The function atan2 is available in most programming languages (C, C++, PHP, Java, ...) and is a version of $\arctan(x)$ (or atan in those languages). Note that $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan(\theta)$. If our point is in the first or fourth quadrant, then $\theta = \arctan\left(\frac{y}{x}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Otherwise, $\theta = \arctan\left(\frac{y}{x}\right) + \pi$ (see next example).

Example 209. Find the polar coordinates, with $r \geqslant 0$ and $\theta \in [0, 2\pi)$ of $(5, 5)$ and $(-5, -5)$.

Solution. The polar coordinates of $(5, 5)$ are $r = 2\sqrt{5}$ and $\theta = \frac{\pi}{4}$.

The polar coordinates of $(-5, -5)$ are $r = 2\sqrt{5}$ and $\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$.

Note. $(5, 5)$ is in the first quadrant and $\theta = \arctan\left(\frac{y}{x}\right) = \arctan(1) = \frac{\pi}{4}$. On the other hand, $(-5, -5)$ is in the third quadrant, and so $\theta = \arctan\left(\frac{y}{x}\right) + \pi = \arctan(1) + \pi = \frac{5\pi}{4}$. [atan2 allows us to avoid this distinction.]

Example 210. Describe a circle around the origin with radius 3 using cartesian and polar coordinates.

Solution. Using cartesian coordinates, the circle is described by $x^2 + y^2 = 3^2$.

Using polar coordinates, the circle is described by the even simpler equation $r = 3$.

Note. In this case, both coordinate equations are easy to see directly. We can, however, convert any equation in cartesian coordinates to polar coordinates by substituting $x = r\cos\theta$ and $y = r\sin\theta$. In our case, we would go from $x^2 + y^2 = 3^2$ to $(r\cos\theta)^2 + (r\sin\theta)^2 = 3^2$, which simplifies to $r^2 = 9$ or $r = 3$ (if we work with $r \geqslant 0$).

Example 211. Which shape is described by $1 \leqslant r \leqslant 3$, $0 \leqslant \theta \leqslant \frac{\pi}{4}$?

Solution. The inequality $1 \leqslant r \leqslant 3$ describes an annulus (shaped like a CD: a disk with a hole).

The inequality $0 \leqslant \theta \leqslant \frac{\pi}{4}$ describes a cone.

Now, put these two together...

Example 212. Describe the y-axis using polar coordinates.

Solution. $\theta = \pm\frac{\pi}{2}$

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