Example 201. Find a trig identity for $\sin(2x)$.

Solution. We will use $e^{2ix} = (e^{ix})^2$ to find such a trig identity. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{iy} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts, we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
[Note that this is just the important special case $x = y$ of our final example from last time.]

Example 202. Which trig identity hides behind $e^{ix}e^{-ix} = 1$?

Solution. Note that

$$\begin{aligned} e^{ix}e^{-ix} &= [\cos(x) + i\sin(x)][\cos(-x) + i\sin(-x)] = [\cos(x) + i\sin(x)][\cos(x) - i\sin(x)] \\ &= \cos^2 x + \sin^2 x. \end{aligned}$$

Hence, $e^{ix}e^{-ix} = 1$ translates into Pythagoras' identity $\cos^2 x + \sin^2 x = 1$.

The **hyperbolic cosine** and **sine** are $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

The remaining hyperbolic trigonometric functions are built from these two as expected.
For instance, $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$.

Example 203. Verify that $\cosh'(x) = \sinh(x)$ and $\sinh'(x) = \cosh(x)$.

Example 204. Determine the Taylor series for the hyperbolic cosine $\cosh(x)$ at $x = 0$.

Solution. Using $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$, we find that

$$\cosh(x) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{1}{2} + \frac{1}{2}(-1)^n\right)\frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

[In the last step, we used that $\frac{1}{2} + \frac{1}{2}(-1)^n = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$ to simplify the sum.]

Solution. Alternatively, we can proceeded from scratch: the derivatives of $f(x) = \cosh(x)$ cycle through $\cosh(x), \sinh(x), \cosh(x), \sinh(x), \ldots$. In particular, $f^{(2n)}(0) = 1$ and $f^{(2n+1)}(0) = 0$.
Therefore, the Taylor series of $f(x) = \cosh(x)$ at $x = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}.$$

Example 205. Observe that $\cosh(x) = \cos(ix)$.

Solution. Do it! Replace x with ix in the Taylor series $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

Remark 206. It follows right from the definition that $e^x = \cosh(x) + \sinh(x)$.
This is a "cheap" version of Euler's identity $e^{ix} = \cos(x) + i\sin(x)$.
In both cases, e^x and e^{ix} are broken up into their even part and odd part.

Definition 207. The **polar coordinates** (r, θ) represent the point $(x, y) = r(\cos\theta, \sin\theta)$.

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