Review 197. Taylor series, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

Note. The series for e^x and $\cos x$ converge for all x (radius of convergence ∞).

Note. $\cos x$ is an even function, and so its Taylor series only includes the even terms x^{2n}.

Example 198. Determine the Taylor series of $f(x) = \sin(x)$ at $x = 0$.

Solution. Integrate $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ to find $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} + C$. Clearly, $C = 0$. (Why?)

Solution. Differentiate $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ to find $-\sin x = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} x^{2n-1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n+1}$.

[In the last step, we replaced n in the summation with $n+1$: note that $\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+1}$.]

We again conclude that $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$.

Solution. The derivatives of $f(x)$ cycle through $\sin(x), \cos(x), -\sin(x), -\cos(x), \ldots$.

In particular, the values $f^{(n)}(0)$ cycle through $0, 1, 0, -1, \ldots$. That is, $f^{(2n)}(0) = 0$ and $f^{(2n+1)}(0) = (-1)^n$.

Therefore, the Taylor series of $f(x) = \sin(x)$ at $x = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Theorem 199. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$.

Why?

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \ldots \\ \cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots \\ \sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \end{aligned}$$

More formally, $e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

Example 200. Which trig identity hides behind $e^{i(x+y)} = e^{ix} e^{iy}$?

Solution. We observe that

$$\begin{aligned} e^{i(x+y)} &= \cos(x+y) + i\sin(x+y) \\ e^{ix} e^{iy} &= [\cos(x) + i\sin(x)][\cos(y) + i\sin(y)] \\ &= \cos(x)\cos(y) - \sin(x)\sin(y) + i(\cos(x)\sin(y) + \sin(x)\cos(y)). \end{aligned}$$

Comparing real and imaginary parts, we conclude that

- $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and
- $\sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y)$.

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