Taylor series

Review 190. Let $f(x)$ be a function. What is the best linear approximation to $f(x)$ at $x=a$.

Solution. The best linear approximation is $f(a)+f'(a)(x-a)$.
[Of course, assuming that $f(x)$ is differentiable at $x=a$.]

The **Taylor series** of $f(x)$ at $x=a$ is the series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\ldots$$

If $f(x)$ can be written as a power series about $x=a$, then $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ for all x such that $|x-a|<R$ (with R the radius of convergence).

- The Taylor at $x=0$ is sometimes also called the **Maclaurin series** of $f(x)$.
- The functions we meet in practice can usually be written as power series, at least about most points (and it usually is not difficult to tell if a special point might be problematic).
  A theoretical guarantee is given by Taylor's formula, which says that

$$f(x)=\sum_{n=0}^{N}\frac{f^{(n)}(a)}{n!}(x-a)^n+R_N(x),\quad \text{with } R_N(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

  for some c between a and x. If $R_N(x)\to 0$ as $N\to\infty$, then $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$.

Example 191. Determine the Taylor series of $f(x)=e^x$ at $x=0$.

Solution. All derivatives of $f(x)$ are e^x. In particular, the values $f^{(n)}(0)=1$ for all n.

Therefore, the Taylor series of $f(x)=e^x$ at $x=0$ is $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$.

Note. Assuming that e^x can be written as a power series at $x=0$, we conclude that $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

[This assumption is justified, because e^x satisfies the simple differential equation $y'=y$. Recall what we did in Example 174! Another way to justify this, is to use Taylor's formula above.]

Example 192. Determine the Taylor series of $f(x)=\cos(x)$ at $x=0$.

Solution. The derivatives of $f(x)$ cycle through $\cos(x),-\sin(x),-\cos(x),\sin(x),\ldots$.

In particular, the values $f^{(n)}(0)$ cycle through $1,0,-1,0,\ldots$. That is, $f^{(2n)}(0)=(-1)^n$ and $f^{(2n+1)}(0)=0$.
Therefore, the Taylor series of $f(x)=\cos(x)$ at $x=0$ is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{f^{(2n)}(0)}{(2n)!}x^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}.$$

Note. Assuming that $\cos x$ can be written as a power series at $x=0$, we conclude that $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$.

[Again, this can be justified via a differential equation or Taylor's formula.]