**Example 184.** Does the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge?

**Solution.** Yes, it converges by the alternating series test: $a_n = \frac{1}{n}$ is positive, decreasing, and $\lim_{n\to\infty} a_n = 0$.

**Note.** Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does not **converge absolutely**.

**Example 185.** For which $p$ does the alternating $p$-series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converge?

**Solution.** If $p > 0$, then the series converges by the alternating series test, because $a_n = \frac{1}{n^p}$ is positive, decreasing, and $\lim_{n\to\infty} a_n = 0$. If $p \leqslant 0$, then $\lim_{n\to\infty} \frac{(-1)^n}{n^p}$ is not zero. Therefore, the series diverges.

In summary, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges if and only if $p > 0$.

**Example 186.** For which $p$ does the alternating $p$-series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converge absolutely?

**Solution.** By definition, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges absolutely if and only if $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^p}\right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Since this is just the usual $p$-series, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges absolutely if and only if $p > 1$.

## Complex numbers

Complex numbers are of the form $x + yi$, where $x$ and $y$ are real numbers, and $i$ (the **imaginary unit**) is a new kind of number characterized by the property that $i^2 = -1$ (or, $i = \sqrt{-1}$).

The next two examples show that we can add, multiply and divide by complex numbers without the need to introduce further new kinds of numbers.

**Example 187.** Simplify $(1+2i) + (2+3i)$ and $(1+2i)(2+3i)$.

**Solution.** $(1+2i) + (2+3i) = 3 + 5i$ and $(1+2i)(2+3i) = 2 + 3i + 4i + 6i^2 = -4 + 7i$

**Example 188.** Simplify $\frac{1}{3-4i}$.

**Solution.** $\frac{1}{3-4i} = \frac{3+4i}{(3-4i)(3+4i)} = \frac{3+4i}{3^2+4^2} = \frac{3}{25} + \frac{4}{25}i$

In fact, power series allow us to make sense out things like $e^{1+2i}$, because $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$ is built just from addition and multiplication.

**Remark 189.** **(advanced!)** Why does the power series $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ only converge for $|x| \leqslant 1$ despite $\arctan(x)$ being differentiable for all $x$?

[Whereas, for contrast, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x$.]

The answer lies in the fact that $\arctan(x)$ has a problem at $x = i$ (which we don't see if we restrict to real $x$). This problem is very visible in $\frac{1}{1+x^2}$, which is the derivative of $\arctan(x)$. Since $|i| = 1$, the power series can have at most radius (!) of convergence 1 (which it does).

Armin Straub
straub@southalabama.edu