Review. When does $\sum_{n=1}^{\infty} \frac{x^n+1}{2^n}$ converge? Evaluate it in that case.

Solution. $\sum_{n=1}^{\infty} \frac{x^n+1}{2^n} \overset{\text{go through}}{\underset{\text{the steps!}}{=}} \frac{1}{1-\frac{x}{2}} - 1 + \frac{1}{1-\frac{1}{2}} - 1 = \frac{1}{1-\frac{x}{2}}$ provided that $|x| < 2$.

Let us follow up on Example 177.

Example 181. Find a power series (about $x=0$) for $\arctan(x)$.

Solution. Recall that $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan(x) + C$.

In Example 177, we observed that $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ and that this power series converges if $|x| < 1$.

We now integrate both sides of $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ to find a power series for $\arctan(x)$.

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n} \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C$$

Hence, $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C$. Since $\arctan(0) = 0$, it follows that $C = 0$.

Example 182. Exact interval of convergence in the previous example.

- Since the convergence radius is 1, we know that the series converges for $|x| < 1$, and diverges if $|x| > 1$. We don't yet know whether the series converges for $x = \pm 1$.
  [In other words, the exact interval of convergence is one of $(-1,1)$, $(-1,1]$, $[-1,1)$, $[-1,1]$.]
- For $x=1$, we get the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

  This is an **alternating series** because the terms are alternately positive and negative.

  [If we sum instead the absolute values of the terms, then we get $\sum_{n=0}^{\infty} \frac{1}{2n+1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$, and we know that this series diverges, because it is "half" of the harmonic series.
  We therefore say that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ is not **absolutely convergent**.]

  Due to the alternating series test below, the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges ($a_n = \frac{1}{2n+1}$ is positive, decreasing and converges to 0).

  Since $\arctan(1) = \frac{\pi}{4}$, we conclude that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$.

  [Series, like this one, that are convergent but not absolutely convergent are called **conditionally convergent**.
  Extra care is required when working with such sequences.]
- Same story for $x = -1$ (do it!). Our conclusion is that the exact interval of convergence is $[-1,1]$.

Theorem 183. (Alternating series test) If a_n is a positive, decreasing sequence with $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=N}^{\infty} (-1)^n a_n$ converges.

Proof by picture!

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