Review. power series, radius of convergence

Example 170. Determine for which x the following series converge. Evaluate these series under this condition (they are geometric). What is their radius of convergence?

(a) $\sum_{n=0}^{\infty} 3^n x^n$ This is a power series about $x=0$.

Solution. $\sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty} (3x)^n$ converges if and only if $|3x|<1$ (we know that because it is geometric).

Equivalently, the series converges if $|x|<1/3$.

Therefore, the convergence radius of this power series is $1/3$.

If $|x|<1/3$, then $\sum_{n=0}^{\infty} 3^n x^n = \frac{1}{1-3x}$ (again, because the series is geometric).

(b) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{5^n}$ This is a power series about $x=2$.

Solution. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{x-2}{5}\right)^n$ converges if and only if $\left|\frac{x-2}{5}\right|<1$.

Equivalently, the series converges if $|x-2|<5$. [This is the same as saying $x\in(-3,7)$.]

Therefore, the convergence radius of this power series is 5.

If $|x-2|<5$, then $\sum_{n=0}^{\infty} \frac{(x-2)^n}{5^n} = \frac{1}{1-\frac{x-2}{5}} = \frac{5}{7-x}$.

Example 171. What is the radius of convergence of the following power series?

(a) $\sum_{n=0}^{\infty} n! x^n$

Solution. We apply the ratio test with $a_n = n! x^n$.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)! x^{n+1}}{n! x^n}\right| = |x|\,(n+1) \to \infty$ as $n\to\infty$ (unless $|x|=0$)

The ratio test implies that $\sum_{n=1}^{\infty} n! x^n$ diverges if $|x|>0$. Radius of convergence is 0.

(b) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Solution. Done this before! The ratio test implies that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x.

Radius of convergence is ∞.

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