Making mischievous mistakes.

$$x^2 = \underbrace{x + x + \ldots + x}_{x \text{ times}} \rightsquigarrow \frac{\mathrm{d}}{\mathrm{d}x} x^2 = \frac{\mathrm{d}}{\mathrm{d}x}(x + x + \ldots + x) \rightsquigarrow 2x = 1 + 1 + \ldots + 1 = x \rightsquigarrow 2 = 1$$

[If you are bothered by the "x times", then note that the above can be written as $x^2 = xy$ with $y = x$. Differentiating both sides, we then have $2x = y$ or $2x = x$, and so $2 = 1$. Can you see where we messed up?]

Lesson: When taking derivatives like $\frac{\mathrm{d}}{\mathrm{d}x} xy$, we need to take into account that y might depend on x (here it does: $y = x$). The product rule then gives us $\frac{\mathrm{d}}{\mathrm{d}x}(xy) = y + x\frac{\mathrm{d}y}{\mathrm{d}x} = y + x = 2x$ and the result matches what we know is correct.

Making more mischievous mistakes.

$$0 = (1-1) + (1-1) + (1-1) + \ldots = 1 - 1 + 1 - 1 + 1 - 1 + \ldots = 1 + (-1+1) + (-1+1) + \ldots = 1 + 0 + 0 \ldots = 1$$

Lesson: Divergent series, like $\sum_{n=0}^{\infty} (-1)^n$, don't conform to our usual laws. It is similar to $\infty = \infty + 1$, so $0 = 1$.

However, if a series converges, then we can work with it. The following argument for evaluating the geometric series is valid provided that the series converges (which we know is if $|x| < 1$).

$$S = \sum_{n=0}^{\infty} x^n \rightsquigarrow xS = \sum_{n=0}^{\infty} x^{n+1} = \sum_{n=1}^{\infty} x^n = S - 1 \rightsquigarrow S = \frac{1}{1-x}$$

Theorem 168. Every power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a **radius of convergence** R, meaning:

(a) if $R = 0$, then the series converges only for $x = a$,

(b) if $0 < R < \infty$, then the series converges for all x such that $|x - a| < R$ but diverges if $|x - a| > R$ (in other words, R is as large as possible),

(c) if $R = \infty$, then the series converges for all x.

Note that, if $0 < R < \infty$, no general statement can be made for the case $|x - a| = R$.

The exact **interval of convergence** can be $(a - R, a + R)$ or $[a - R, a + R)$ or $(a - R, a + R]$ or $[a - R, a + R]$.

Example 169. Determine for which x the following series converge. What is their radius of convergence?

(a) $\sum_{n=1}^{\infty} n x^n$ — This is a power series about $x = 0$.

Solution. We apply the ratio test with $a_n = n x^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)x^{n+1}}{n x^n}\right| = |x| \frac{n+1}{n} \to |x| \text{ as } n \to \infty$$

The ratio test implies that $\sum_{n=1}^{\infty} n x^n$ converges if $|x| < 1$. Radius of convergence is 1.

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2} (x-3)^n$ — This is a power series about $x = 3$.

Solution. We apply the ratio test with $a_n = \frac{2^n}{n^2}(x-3)^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(x-3)^{n+1}}{(n+1)^2} \frac{n^2}{2^n (x-3)^n}\right| = 2|x-3| \frac{n^2}{(n+1)^2} \to 2|x-3| \text{ as } n \to \infty$$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{2^n}{n^2}(x-3)^n$ converges if $|x - 3| < 1/2$. Radius of convergence is $1/2$.

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