Review 145. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (in both cases, $\lim_{n\to\infty} a_n = 0$).

Example 146. For what values of p does $\int_1^{\infty} \frac{dx}{x^p}$ converge?

Solution. For $p \neq 1$, we have $\int_1^{\infty} \frac{dx}{x^p} = \left[\frac{1}{-p+1} x^{-p+1}\right]_1^{\infty}$.

If $p > 1$, then $\lim_{x\to\infty} x^{-p+1} = \lim_{x\to\infty} \frac{1}{x^{p-1}} = 0$, and we find that the integral converges.

If $p < 1$, then $\lim_{x\to\infty} x^{-p+1} = \infty$, and we find that the integral diverges.

We are missing only the case $p = 1$: in that case, $\int_1^{\infty} \frac{1}{x} dx = \left[\log x\right]_1^{\infty}$ diverges because $\lim_{x\to\infty} \log(x) = \infty$.

To summarize: $\int_1^{\infty} \frac{dx}{x^p}$ converges if and only if $p > 1$.

Example 147. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-**series**. It converges if and only if $p > 1$. Why?

Solution. This follows from the integral comparison test, and because $\int_1^{\infty} \frac{dx}{x^p}$ converges if and only if $p > 1$.

Remark 148. If $p > 1$, then $\int_1^{\infty} \frac{dx}{x^p} = \left[\frac{1}{-p+1} x^{-p+1}\right]_1^{\infty} = \frac{1}{p-1}$.

However, we cannot evaluate $\sum_{n=1}^{\infty} \frac{1}{n^p}$ in any easy way.

- Comparison with the integral only produces inequalities: for instance, $\sum_{n=1}^{\infty} \frac{1}{n^2} > \int_1^{\infty} \frac{dx}{x^2} = 1$.

  [Note that this inequality is worthless because it is obvious that $1 + \frac{1}{4} + \frac{1}{9} + \ldots > 1$.]

- The values of p-series are very mysterious to this day:
  - Euler proved (and became famous for doing so) that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$, …
  - However, we know almost nothing about the $\sum_{n=1}^{\infty} \frac{1}{n^3}$, $\sum_{n=1}^{\infty} \frac{1}{n^5}$, …

    [To give you an idea how little we know: Apéry became famous for showing in 1978 that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is not a rational number. We still don't know whether it is π^3 times a rational number.
    By the way, the curious numbers from Example 122 were fundamental to Apéry's proof.]

Example 149. Determine whether the following series converge or diverge.

(a) $\sum_{n=0}^{\infty} \frac{1}{2n+1}$

Your final answer should be that this series diverges.

(b) $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$

Your final answer should be that this series converges.

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