Example 141. Determine whether the following series converge or diverge. If possible, determine their value.

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\log n}$

Solution. This series diverges by Theorem 139 because $\lim_{n\to\infty} \frac{\sqrt{n}}{\log n}$ is not zero (in fact, that limit is ∞).

(b) $\sum_{n=1}^{\infty} \frac{3^n+5^n}{7^n}$

Solution. $\sum_{n=1}^{\infty} \frac{3^n+5^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{3}{7}\right)^n + \sum_{n=1}^{\infty} \left(\frac{5}{7}\right)^n$. Since $\left|\frac{3}{7}\right| < 1$ and $\left|\frac{5}{7}\right| < 1$, the series converges. Find its value!

(c) $\sum_{n=1}^{\infty} \frac{3^n+7^n}{5^n}$

Solution. This series diverges. (Why?!)

Example 142. The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges. Why?

Solution. Note that $\lim_{n\to\infty} \frac{1}{n} = 0$, so we cannot directly use our test for divergence coming out of Theorem 139. However, we can combine terms as follows to see the divergence:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geqslant \frac{1}{4}+\frac{1}{4}=\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geqslant \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{\geqslant \frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{1}{2}} + \ldots$$

Solution. Here is another way to see that the harmonic series diverges. A quick plot reveals that

$$\sum_{n=1}^{M} \frac{1}{n} \geqslant \int_1^M \frac{1}{x} \mathrm{d}x.$$

(A similar plot is Figure 9.11 (a) in the book.) But we already know (or can quickly check; do it!) that, in the limit $M \to \infty$, the integral $\int_1^{\infty} \frac{1}{x} \mathrm{d}x$ diverges. It follows, by comparison, that the harmonic series diverges, too.

Theorem 143. (Integral comparison test) Suppose that $f(x)$ is a positive, continuous, decreasing function for $x \geqslant N$. Then:

$$\sum_{n=N}^{\infty} f(n) \text{ converges} \iff \int_N^{\infty} f(x) \mathrm{d}x \text{ converges}$$

In other words, the series and integral both converge or both diverge.
Warning: if they converge, of course, the values of the series and the integral are going to be different!

Example 144. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$ converges.

[It is considerably more difficult to show that, in fact, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.]

Solution. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges if and only if the integral $\int_1^{\infty} \frac{1}{x^2} \mathrm{d}x$ converges.

Since $\int_1^{\infty} \frac{1}{x^2} \mathrm{d}x = \left[-\frac{1}{x}\right]_1^{\infty} = 0 - (-1) = 1$, the integral converges, and so the series converges as well.

Armin Straub
straub@southalabama.edu