Example 137. Write the series $e^{2x}+e^{3x}+e^{4x}+\ldots$ using Σ-notation and evaluate it.

Solution. $e^{2x}+e^{3x}+e^{4x}+\ldots=\sum_{n=2}^{\infty} e^{nx}=\sum_{n=0}^{\infty} e^{nx}-1-e^x=\frac{1}{1-e^x}-1-e^x$ provided that $|e^x|<1$ (or, equivalently, $x<0$).

[or: $e^{2x}+e^{3x}+e^{4x}+\ldots=e^{2x}(1+e^x+e^{2x}+\ldots)=e^{2x}\sum_{n=0}^{\infty} e^{nx}=\frac{e^{2x}}{1-e^x}$. Check that this is the same!]

Example 138. Express the number $2.313131\ldots$ as a rational number.

Solution. Your final answer should be $2+\frac{31}{99}=\frac{229}{99}$.

This example plus the last one from previous class teach us something fundamental about numbers:

> Rational numbers are precisely those numbers which have a finite (like 1.5) or repeating (like $2.313131\ldots$) decimal expansion.
>
> Moreover, there is some ambiguity because finite decimals, like 1.5, can also be written in the repeating fashion $1.5=1.4999\ldots$.

As a consequence, irrational numbers like $\sqrt{2}$ or π never have a repeating decimal expansion.

Recall that the improper integral $\int_N^{\infty} f(x)\mathrm{d}x$ converges if and only if the limit $\lim_{M\to\infty}\int_N^M f(x)\mathrm{d}x$ exists. Likewise, $\sum_{n=N}^{\infty} a_n$ converges if and only if the limit $\lim_{M\to\infty}\sum_{n=N}^{M} a_n$ exists.

> **Theorem 139.** For the series $\sum_{n=N}^{\infty} a_n$ to converge it is necessary that $\lim_{n\to\infty} a_n=0$.

This is simply saying that the only hope to be able to add infinitely many things (and get something finite) is if these things are very small.

Hence, we have a first simple test for divergence: if $\lim_{n\to\infty} a_n$ is not 0 (or DNE), then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 140. Show that the following series all diverge.

(a) $\sum_{n=1}^{\infty}\frac{n}{\log(n)}$

(b) $\sum_{n=1}^{\infty}\frac{3^n}{2^n}$

(c) $\sum_{n=1}^{\infty}(-1)^n$

(d) $\sum_{n=1}^{\infty}\frac{n^2}{3n^2+7}$

A word of caution: Theorem 139 only gives a necessary condition. It is not sufficient!

For instance, as we will see next time, the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges although $\lim_{n\to\infty}\frac{1}{n}=0$.

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