Example 134. Compute the following series (or state that it diverges):

(a) $\sum_{n=0}^{\infty} \frac{5}{3^n} =$

Your final answer should be $\frac{15}{2}$.

(b) $\sum_{n=2}^{\infty} 3 \cdot 4^{-n} =$

Your final answer should be $\frac{1}{4}$.

(c) $\sum_{n=0}^{\infty} \left(\frac{7}{2^n} - \frac{3^n}{5^n} \right) =$

Your final answer should be $\frac{23}{2}$.

(d) $\sum_{n=0}^{\infty} \frac{5^n}{3^n} =$

This series doesn't converge.

(e) $\sum_{n=0}^{\infty} (-1)^n x^{2n} =$

Your final answer should be $\frac{1}{1+x^2}$ under the condition that $|-x^2| < 1$ (which is the same as $|x| < 1$). If this condition is not true, then the series diverges.

Remark 135. The very last example illustrates an important point. Namely, it shows that there is a novel way to think about (and get our hands on) functions like $\frac{1}{1+x^2}$.

[Recall that we care about this function in particular, because it was a building block in partial fractions. For instance, we know that its antiderivative is $\arctan(x)$.]

This is the main reason why we are learning about series in a course that focuses on functions!

We will see that it is very convenient to work with series representing functions: they can be differentiated and integrated, and give us an opportunity to work with functions that cannot be written in terms of the "usual" functions.

Example 136. Is $0.999999... = 1$?

[One indication (which does not rely on the notions of limits or series) that the answer should be yes, is that $1/3 = 0.333333...$ and so multiplying with 3 should get us back to 1.]

Solution. Note that, by definition, $0.999999... = 0.9 + 0.09 + 0.009 + ... = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + ...$
So, to give a definitive answer, we need to compute this infinite sum:

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + ... = \sum_{n=1}^{\infty} \frac{9}{10^n} = 9\left(\sum_{n=0}^{\infty} \frac{1}{10^n} - 1 \right) = \frac{9}{1 - \frac{1}{10}} - 9 = 1$$

Solution. Another way to think about $0.999999...$ as the limit of the sequence $0.9,\ 0.99,\ 0.999,\ 0.9999,\ ....$ Read again the definition of a limit in Definition 119, and conclude that this sequence converges to 1!

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