Review 125. Indeterminate forms are "$\frac{\infty}{\infty}$", "$\frac{0}{0}$", "$0\cdot\infty$", "∞^0", "1^∞" "0^0". By taking $\log$ in the last three cases, we can always write these as "$\frac{\infty}{\infty}$" or "$\frac{0}{0}$", so that we can again apply L'Hospital.

We can "see" the limits $\lim_{n\to\infty}\frac{3n^2+7n-8}{8n^3+n+1}=0$ or $\lim_{n\to\infty}\frac{3n^2+7n-8}{8n^2+n+1}=\frac{3}{8}$.

Of course, we also know how to apply, for instance, L'Hospital to find these limits.

Example 126. $\lim_{n\to\infty}\left(\frac{3}{n}\right)^{1/n}=$

Solution. If $\lim_{n\to\infty}\left(\frac{3}{n}\right)^{1/n}=L$, then $\lim_{n\to\infty}\log\left(\left(\frac{3}{n}\right)^{1/n}\right)=\log(L)$. We can compute the latter as

$$\lim_{n\to\infty}\log\left(\left(\frac{3}{n}\right)^{1/n}\right)=\lim_{n\to\infty}\frac{\log\left(\frac{3}{n}\right)}{n}=\lim_{n\to\infty}\frac{\log(3)-\log(n)}{n}\underset{\text{L'Hospital}}{\overset{\text{“}\frac{\infty}{\infty}\text{”}}{=}}\lim_{n\to\infty}\frac{-\frac{1}{n}}{1}=0.$$

From $\log(L)=0$ we conclude $L=e^0=1$. So, $\lim_{n\to\infty}\left(\frac{3}{n}\right)^{1/n}=1$.

Example 127. $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$ (for any x)

Solution. Apply $\log$ to the sequence, then apply L'Hospital. You should find that $\lim_{n\to\infty}\log\left(\left(1+\frac{x}{n}\right)^n\right)=x$. Finally, undo the $\log$.

Example 128. What is the limit of the sequence $\sqrt{6}$, $\sqrt{6+\sqrt{6}}$, $\sqrt{6+\sqrt{6+\sqrt{6}}}$, ...?

Solution. This sequence $\{a_n\}$ is defined recursively: $a_1=\sqrt{6}$ and $a_n=\sqrt{6+a_{n-1}}$ for $n\geqslant 2$. Computing the first few terms numerically, it seems that $\lim_{n\to\infty}a_n$ exists and is about 3.

- Suppose that $\lim_{n\to\infty}a_n=L$. Taking the limit of both sides of $a_n=\sqrt{6+a_{n-1}}$, we get

$$L=\lim_{n\to\infty}a_n=\lim_{n\to\infty}\sqrt{6+a_{n-1}}=\sqrt{6+L}.$$

- Writing $L=\sqrt{6+L}$ as $L^2=6+L$ and solving this quadratic equation shows that $L=\frac{1\pm\sqrt{25}}{2}$.
- Since $\frac{1-\sqrt{25}}{2}=-2$ (and our sequence is positive), the limit (if it exists) has to be $L=\frac{1+\sqrt{25}}{2}=3$.

Example 129. What is the limit of the sequence $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{8}{5}$, $\frac{13}{8}$, $\frac{21}{13}$, $\frac{34}{21}$, ...?

Hints. Recall that $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ are the **Fibonacci numbers** $\{F_n\}$.
They are defined **recursively**: $F_1=1$, $F_2=1$ and $F_n=F_{n-1}+F_{n-2}$ for $n\geqslant 3$.

- Our sequence are quotients of Fibonacci numbers $\{a_n\}$ with $a_n=\frac{F_{n+1}}{F_n}$.
- Take $F_{n+1}=F_n+F_{n-1}$ and divide both sides by F_n to get the recursive relation $a_n=1+\frac{1}{a_{n-1}}$.
- Now, suppose our sequence converges and that $\lim_{n\to\infty}a_n=L$.
  Proceed as in the previous example and take the limit of both sides of $a_n=1+\frac{1}{a_{n-1}}$.
  ...
- Once, you have determined the limit, compare it numerically with our sequence.
  Are the terms indeed approaching the value you found?

Armin Straub
straub@southalabama.edu