Example 102. $\int \sin^{\lambda}(x)\cos(x)\,dx =$ (with $\lambda \neq -1$)

Solution. Again, substitute $u = \sin(x)$, because $du = \cos(x)\,dx$, to get

$$\int \sin^{\lambda}(x)\cos(x)\,dx = \int u^{\lambda} du = \frac{1}{\lambda+1} u^{\lambda+1} + C = \frac{\sin^{\lambda+1}(x)}{\lambda+1} + C.$$

Example 103. $\int \sin^{\lambda}(x)\cos^3(x)\,dx =$ (with $\lambda \neq -1, -3$)

Solution. Again, substitute $u = \sin(x)$, because $du = \cos(x)\,dx$, to get

$$\int \sin^{\lambda}(x)\cos^3(x)\,dx = \int u^{\lambda}(1-u^2)du = \frac{u^{\lambda+1}}{\lambda+1} - \frac{u^{\lambda+3}}{\lambda+3} + C = \frac{\sin^{\lambda+1}(x)}{\lambda+1} - \frac{\sin^{\lambda+3}(x)}{\lambda+3} + C.$$

Extrapolating this strategy, we can integrate the following products of trigonometric function:

- $\int \sin^{\lambda}(x)\cos^m(x)\,dx$, with m odd, can be evaluated by substituting $u = \sin(x)$.
- $\int \sin^m(x)\cos^{\lambda}(x)\,dx$, with m odd, can be evaluated by substituting $u = \cos(x)$.
- $\int \sin^m(x)\cos^n(x)\,dx$, with both m, n even, can be reduced via

$$\sin^2(x) = \frac{1-\cos(2x)}{2}, \quad \cos^2(x) = \frac{1+\cos(2x)}{2}.$$

[Then, multiply out the integrand. The resulting integrals are simpler, and we (recursively) apply our strategy to each of them (if the $2x$ bothers you, substitute $u = 2x$).]

Exponents may also be negative (in the next example, we integrate $[\sin(x)]^1[\cos(x)]^{-1}$).

Example 104. $\int \tan(x)\,dx =$

Solution. $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}dx$, so we substitute $u = \cos(x)$ (then $du = -\sin(x)$) to get

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos(x)| + C = \ln|\sec(x)| + C.$$

Solution. For some exercise in substituting, substitute $u = \sin(x)$ instead!
[Of course, your final answer should be equivalent.]

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